CHAPTER 2

Preliminaries

In this chapter, we begin with basic knowledge of semigroup theory that will be used throughout this thesis.

2.1 Elementary definitions

Let S be a nonempty set and $\cdot$ a binary operation on S. We say that S is a *semigroup* if it satisfies the associative law, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$. In this note, we use ab instead of $a \cdot b$. A nonempty subset A of S is called a *subsemigorup* of S if $ab \in A$ for all $a, b \in A$. For nonempty subsets A, B of S, define $AB = \{ab \mid a \in A, b \in B\}$.

The following definitions are taken from [3] and [18].

Definition 2.1.1. An element a of a semigroup S is called an *idempotent element* if $a = a^2$. The set of all idempotent elements of a semigroup S is denoted by $E(S)$.

Definition 2.1.2. An element a of a semigroup S is called a *regular element* if $a = asa$ for some $s \in S$. The set of all regular elements of a semigroup S is denoted by $Reg(S)$.

Note that every idempotent element is a regular element.

Definition 2.1.3. A semigroup S is *regular* if all elements of S are regular.

Definition 2.1.4. Let S be a semigroup and A a nonempty subset of S. Then,

- A is called a *right ideal* of S if $AS \subseteq A$.
- A is called a *left ideal* of S if $SA \subseteq A$.
- A is called an *ideal* of S if A is both left and right ideal of S.

An ideal A [resp. right ideal, left ideal] of S is called *proper ideal* [resp. *proper right ideal, proper left ideal*] of S if $A \neq S$.

Definition 2.1.5. An element a of a semigroups S is called a *zero element* if $as = sa = a$ for all $s \in S$. In this thesis, we denote by 0 a zero element of S.

Definition 2.1.6. Let S be a semigroup and 0 a zero element of S. Then,

- S is called a *right simple semigroup* if it has no proper right ideals.

- S is called a *left simple semigroup* if it has no proper left ideals.

- S is called *right* 0*-simple semigroup* if $S \neq \{0\}$ and if $\{0\}$ and S are only right ideals of S.

- S is called *left* 0*-simple semigroup* if $S \neq \{0\}$ and if $\{0\}$ and S are only left ideals of S.

2.2 BQ-semigroups

In this section, we present the concept of quasi-ideals and bi-ideals which are generalization of left ideals, right ideals and ideals.

Definition 2.2.1. Let S be a semigroup and A a subsemigroup of S. Then,

- A is called a *bi-ideal* of S if $ASA \subseteq A$.

- A is called a *quasi-ideal* of S if $AS \cap SA \subseteq A$.

In general, it is known that every quasi-ideal A of a semigroup S is a bi-ideal of S because $ASA \subseteq AS \cap SA \subseteq A$. The following example shows that there is a bi-ideal which is not a quasi-ideal.

Example 2.2.2. (see [17].) Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a, b \in \mathbb{R}^+ \right\}$. Then S is a semigroup under the usual multiplication of matrices. Define $R = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a, b \in \mathbb{R}^+, a < b \right\}$ and $L = \left\{ \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix} \mid p, q \in \mathbb{R}^+, q > 5 \right\}$. Then RL is a bi-ideal of S, but not a quasi-ideal of S.

For a nonempty subset A of a semigroup S, a *bi-ideal of S generated by A*, denoted by $(A)_b$, is given by

$$A \cup A^2 \cup ASA.$$

Also, a *quasi-ideal of S generated by A*, denoted by $(A)_q$, is given by

$$A \cup (AS \cap SA).$$

If A is a singleton, $A = \{a\}$, we call a *principal bi-ideal of S generated by a*, denoted by $(a)_b$, and *principal quasi-ideal of S generated by a*, denoted by $(a)_q$, respectively. The following proposition is well known.

Proposition 2.2.3. *$(A)_b \subseteq (A)_q$ for any nonempty subset A of a semigroup S.*

Theorem 2.2.4. (see [15].) *Let S be a semigroup, $A \subseteq Reg(S)$ and $\emptyset \neq B \subseteq S$. If $(B)_b = (B)_q$, then $(A \cup B)_b = (A \cup B)_q$.*

A semigroup whose set of bi-ideals and quasi-ideals coincide is called a BQ-semigroup. It is well known that every regular semigroup is a BQ-semigroup. In many papers, the authors characterized BQ-semigroups, as follows.

Theorem 2.2.5. (see [2].) *A semigroup S is a BQ-semigroup if and only if*

$$(\{x, y\})_b = (\{x, y\})_q$$

for all $x, y \in S$.

Theorem 2.2.6. (see [10].) *Let S be a semigroup. If S is a right [left] simple semigroup or a right [left] 0-simple semigroup, then S is a BQ-semigroup.*

As in [1, 13], the generalization of bi-ideals and quasi-ideals was studied as shown in the following definitions.

Definition 2.2.7. Let S be a semigroup, A a subsemigroup of S and m, n nonnegative integers.

- A is called an (m, n)*-ideal* of S if $A^m S A^n \subseteq A$.
- A is called an (m, n)*-quasi-ideal* of S if $A^m S \cap S A^n \subseteq A$.

Here, $A^0 S = S = S A^0$.

Proposition 2.2.8. (see [13].) *Every (m, n)-quasi-ideal of a semigroup S is a (m, n)-ideal of S.*

For a nonempty subset A of a semigroup S, an (m, n)*-ideal of S generated by A*, denoted by $(A)_{(m,n)}$, is given by

$$\bigcup_{i=1}^{m+n} A^i \cup A^m S A^n.$$

Also, an (m, n)*-quasi-ideal of S generated by A*, denoted by $(A)_{q(m,n)}$, is given by

$$\bigcup_{i=1}^{max\{m,n\}} A^i \cup (A^m S \cap S A^n).$$

Note that $(A)_b = (A)_{(1,1)}$ and $(A)_q = (A)_{q(1,1)}$, respectively. If A is a singleton, $A = \{a\}$, we call a *principal* (m, n)*-ideal of S generated by a*, denoted by $(a)_{(m,n)}$, and *principal* (m, n)*-quasi-ideal of S generated by a*, denoted by $(a)_{q(m,n)}$, respectively. The following proposition is obtained by Proposition 2.2.8

Proposition 2.2.9. *Let m, n be nonnegative integers. Then $(A)_{(m,n)} \subseteq (A)_{q(m,n)}$ for any nonempty subset A of a semigroup S.*

2.3 Some equivalence relations on BQ-semigroups

In many papers, equivalence relations on semigroups have been studied, as follows.

Definition 2.3.1. (see [6].) Let S be a semigroup. For $a, b \in S$, we write

1. $a\mathcal{L}b$ if $(a)_{(0,1)} = (b)_{(0,1)}$,
2. $a\mathcal{R}b$ if $(a)_{(1,0)} = (b)_{(1,0)}$,
3. $a\mathcal{H}b$ if $a\mathcal{L}b$ and $a\mathcal{R}b$.

Definition 2.3.2. (see [10].) Let S be a semigroup. For $a, b \in S$, we write $a\mathcal{B}b$ if

1. $a = b$; or
2. $a = bvb$ and $b = aua$ for some $u, v \in S$.

Moreover, we denote the $\mathcal{B}$-class containing a by $\mathcal{B}(a)$.

Proposition 2.3.3. (see [10].) *Let S be a semigroup and $a, b \in S$. Then $a\mathcal{B}b$ if and only if $(a)_b = (b)_b$.*

Definition 2.3.4. (see [21].) Let S be a semigroup and m, n nonnegative integers. For $a, b \in S$, we write $a\mathcal{B}_m^n b$ if

1. $a = b$; or
2. $a = b^m v b^n$ and $b = a^m u a^n$ for some $u, v \in S$.

Moreover, we denote the $\mathcal{B}_m^n$-class containing a by $\mathcal{B}_m^n(a)$.

In case $m = n = 1$, we have $\mathcal{B}_1^1 = \mathcal{B}$.

Proposition 2.3.5. (see [21].) *The relation $\mathcal{B}_m^n$ is an equivalence relation. Moreover, $\mathcal{B}_m^n \subseteq \mathcal{B}$.*

Proposition 2.3.6. (see [21].) *Let S be a semigroup and $a, b \in S$. Then $a\mathcal{B}_m^n b$ if and only if $(a)_{(m,n)} = (b)_{(m,n)}$.*

Proposition 2.3.7. (see [21].) *If $a\mathcal{B}_m^n b$, then $a\mathcal{B}b$.*

Definition 2.3.8. (see [18].) Let S be a semigroup. For $a, b \in S$, we write $a\mathcal{Q}b$ if

$$(a)_q = (b)_q.$$

Moreover, we denote the $\mathcal{Q}$-class containing a by $\mathcal{Q}(a)$.

In [18] p. 19, the author shows that $\mathcal{H} = \mathcal{Q}$, where $\mathcal{H}$ is Green's $\mathcal{H}$-relation.

Proposition 2.3.9. (see [10].) *If $a\mathcal{B}b$, then $a\mathcal{Q}b$.*

Theorem 2.3.10. (see [14].) *Let S be a BQ-semigroup. Then $a\mathcal{B}b$ if and only if $a\mathcal{Q}b$.*

2.4 Semigroups of transformations

Let X, Y be nonempty sets such that $Y \subseteq X$. The set of all functions on X with composition $\circ$ is called the *full transformation semigroup on X*, denoted by $T(X)$. For $\alpha \in T(X)$, the *image of α* is given by $X\alpha = \{x \in X \mid x = a\alpha \text{ for some } a \in X\}$ and we use $|X\alpha|$ to denote the cardinality of $X\alpha$. The following sets are subsemigroups of $T(X)$:

$$T(X, Y) = \{\alpha \in T(X) \mid X\alpha \subseteq Y\},$$
$$S(X, Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

Proposition 2.4.1. (see [7].) *$T(X)$ is a regular semigroup.*

Theorem 2.4.2. (see [8,16].) *The semigroup $S(X, Y)$ is regular if and only if $Y = X$ or $|Y| = 1$.*

Theorem 2.4.3. (see [8,16].) *For $\alpha \in S(X, Y)$, $\alpha \in Reg(S(X, Y))$ if and only if $X\alpha \cap Y = Y\alpha$.*

Theorem 2.4.4. (see [15].) *The semigroup $S(X, Y)$ is a BQ-semigroup if and only if one of the following statements holds*

(i) $Y = X$,

(ii) $|Y| = 1$,

(iii) $|X| \leq 3$.