CHAPTER 3

Main Results

Recall that a subsemigroup A of a semigroup S is said to be (m,n)-ideal [resp. (m,n)-quasi-ideal] of S, m,n are nonnegtive integers, if $A^mSA^n \subseteq A$ [resp. $A^mS \cap SA^n \subseteq A$] where $A^0S = S = SA^0$. From the previous chapter, we see that if A is an (m,n)-quasi-ideal of S, then A is an (m,n)-ideal of S. The following example shows that there is an (m,n)-ideal which is not an (m,n)-quasi-ideal.

Example 3.0.1. Let $X = \{1,2,3,4\}$, $Y = \{1,2,3\}$. Define a semigroup $S(X,Y) = \{\alpha : X \to X \mid Y\alpha \subseteq Y\}$ with the composition of function and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}$. Then one can see that

$$(\beta)_{(2,1)} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \right\}$$

is an $(2,1)$-ideal of S, but not an $(2,1)$-quasi-ideal of S.

In this research, we extend the concept of BQ-semigroups to that of (m,n)-BQ-semigroups which is defined as follows.

Definition 3.0.2. Let m,n be nonnegative integers. A semigroup S is called an (m,n)-BQ-semigroup if the set of (m,n)-ideals and (m,n)-quasi-ideals coincide.

In this thesis, we denote the class of all (m,n)-BQ-semigroups by BQ_m^n

3.1 (m,n)-BQ-semigroups

In 1969, Kapp [10] proved that a right [left] simple semigroup and a right [left] 0-simple semigroup are in BQ, see [3, p. 5] and [3, p. 67], respectively. Now, we obtain an analogous result in (m,n)-BQ-semigroups.

Theorem 3.1.1. *Let m,n be nonnegative integers. If S is a right [left] simple semigroup or a right [left] 0-simple semigroup, then $S \in BQ_m^n$.*

Proof. Assume that S is a right simple semigroup. We show that every (m,n)-ideal of S is (m,n)-quasi-ideal of S. Indeed, let A be an (m,n)-ideal of S. Since A^mS is a right ideal of S, by assumption, we have $A^mS = S$. Hence $SA^n = A^mSA^n \subseteq A$, which implies that $A^mS \cap SA^n = S \cap SA^n = SA^n \subseteq A$. Then A is an (m,n)-quasi-ideal of S.

In the case where S is a right 0-simple semigroup, let B be an (m, n)-ideal of S. If $B = \{0\}$, we are done by definition. The case where $B \neq \{0\}$ can be shown as above. Similarly, we can prove that if S is a left simple or a left 0-simple semigroup, then $S \in BQ_m^n$. $\square$

Let m, n be nonnegative integers. In [4], an element $x \in S$ is said to be an (m, n)-*regular element* if $x \in x^m S x^n$ where x^0 is defined by $x^0 y = y x^0 = y$ for all $y \in S$. The set of all (m, n)-regular elements of S is denoted by $Reg_m^n(S)$. In particular, S is said to be (m, n)-*regular* if every element of S is an (m, n)-regular element. Obviously, if $x \in x^m S x^n$, then $x \in xSx$, i.e. $Reg_m^n(S) \subseteq Reg(S)$, where $Reg(S)$ is the set of all regular elements of S. In fact, Lajos [13] has shown that every (m, n)-ideal of a regular semigroup S is an (m, n)-quasi-ideal of S, which leads to the following theorem.

Theorem 3.1.2. *Let m, n be nonnegative integers. Every regular semigroup is an (m, n)-BQ-semigroup.*

In [2], Calais characterized BQ-semigroups: $S \in BQ$ if and only if for all $x, y \in S$, $(\{x, y\})_{(1,1)} = (\{x, y\})_{q(1,1)}$. In this thesis, we generalize this result as follows.

Theorem 3.1.3. *Let S be a semigroup and m, n nonnegative integers. Then the following statements are equivalent:*

(1) *every (m, n)-ideal A of S is an (m, n)-quasi-ideal of S;*

(2) *for every nonempty subset D of S such that $|D| \leq m + n$, the (m, n)-ideal generated by D of S is an (m, n)-quasi-ideal of S.*

Proof. Let A be an (m, n)-ideal of S. Assume that (2) holds. Let $x \in A^m S \cap S A^n$. Then $x = (\Pi_{i=1}^m a_i) s_1 = s_2 (\Pi_{j=1}^n b_j)$ for some $s_1, s_2 \in S$ and some $a_i, b_j \in A$. Let $D = \{a_1, a_2, \ldots, a_m\} \cup \{b_1, b_2, \ldots, b_n\}$. Hence $|D| \leq m + n$ and $(D)_{(m,n)}$ is an (m, n)-quasi-ideal of S by assumption. This implies that $x \in (D)_{(m,n)}^m S \cap S (D)_{(m,n)}^n \subseteq (D)_{(m,n)} \subseteq A$. Thus, A is an (m, n)-quasi-ideal of S. Conversely, if (1) holds, it is easy to see that (2) holds. $\square$

To prove that a semigroup S belongs to BQ_m^n, we have to show that $(A)_{(m,n)} = (A)_{q(m,n)}$ for any nonempty subset A of S such that $|A| \leq m + n$. By Proposition 2.2.9, it suffices to show that $(A)_{q(m,n)} \subseteq (A)_{(m,n)}$. It is obvious that $\bigcup_{i=1}^{max\{m,n\}} A^i \subseteq \bigcup_{i=1}^{m+n} A^i$, so we must show that $A^m S \cap S A^n \subseteq A^m S A^n$. Thus if we want to show that $S \notin BQ_m^n$, we may show that there is an element $x \in A^m S \cap S A^n$ but $x \notin A^m S A^n$. The following theorems are tools for showing that $S \in BQ_m^n$.

Theorem 3.1.4. *Let m, n be nonnegative integers. Every bi-ideal of a regular semigroup is an (m,n)-BQ-semigroup.*

Proof. Let T be a bi-ideal of a regular semigroup S and A an (m,n)-ideal of T. Let $x \in A^mT \cap TA^n$. By the regularity of S, there is $s \in S$ such that $x = xsx$. Then $x = xsx \in A^mTSTA^n \subseteq A^mTA^n \subseteq A$. Therefore, A is an (m,n)-quasi-ideal of T, that is, $T \in BQ_m^n$. $\square$

Theorem 3.1.5. *Let m, n be nonnegative integers. If S is a regular semigroup, then the following statements hold:*

(1) *every right ideal of S is an (m,n)-BQ-semigroup,*

(2) *for any right ideal R of S and left ideal L of S, $R \cap L$ is an (m,n)-BQ-semigroup.*

Proof. Assume that S is a regular semigroup.

(1) Let R be a right ideal of S and A an (m,n)-ideal of R. We show that $A^mR \cap RA^n \subseteq A$. Let $x \in A^mR \cap RA^n$. By assumption, $x \in xSx \subseteq A^mRSRA^n \subseteq A^mRA^n \subseteq A$. So, R is an (m,n)-BQ-semigroup.

(2) Let R be a right ideal of S and L a left ideal of S. We have $\emptyset \neq RL \subseteq R \cap L$. Let B be an (m,n)-ideal of $R \cap L$. For each $y \in B^m(R \cap L) \cap (R \cap L)B^n \subseteq B^mR \cap LB^n$, we obtain by the regularity of S that

$$y \in ySy \subseteq B^mRSLB^n \subseteq B^mRLB^n \subseteq B^m(R \cap L)B^n \subseteq B.$$

Therefore, $R \cap L$ is an (m,n)-BQ-semigroup.

$\square$

Theorem 3.1.6. *Let S be a semigroup and m, n nonnegative integers. If $\emptyset \neq A \subseteq Reg_m^n(S)$, then $(A)_{q(m,n)} = (A)_{(m,n)}$.*

Proof. It suffices to prove that $(A)_{q(m,n)} \subseteq (A)_{(m,n)}$. Assume that $\emptyset \neq A \subseteq Reg_m^n(S)$. Let $x \in (A)_{q(m,n)}$. If $x \in A^i$ for some $i \in \{1, 2, \ldots, max\{m,n\}\}$, then $x \in (A)_{(m,n)}$. Suppose $x \in A^mS \cap SA^n$. If $m = 0$ or $n = 0$, it is clear that $x \in A^mSA^n \subseteq (A)_{(m,n)}$. We assume that $m, n \neq 0$. Then $x = (\Pi_{i=1}^m a_i)s = t(\Pi_{j=1}^n b_j)$ for some $s, t \in S$ and some $a_i, b_j \in A$. If $n = 1$, we have $x = (\Pi_{i=1}^m a_i)s = tb_1$. Since $b_1 \in A \subseteq Reg_m^n(S) \subseteq Reg(S)$, there is $v \in S$ such that $b_1 = b_1vb_1$ and hence $x = tb_1 = tb_1vb_1 = (\Pi_{i=1}^m a_i)svb_1 \in A^mSA = A^mSA^n \subseteq (A)_{(m,n)}$. Now, we suppose that $n > 1$. Since $a_1 \in A$, by assumption, there is $u \in S$ such that $a_1 = a_1^m u a_1^n$ and hence $x = (\Pi_{i=1}^m a_i)s = a_1^m u a_1^n (\Pi_{i=2}^m a_i)s = a_1^m u a_1^{n-1}(\Pi_{i=1}^m a_i)s = a_1^m u a_1^{n-1} t(\Pi_{j=1}^n b_j) \in A^m S A^n$. So $(A)_{q(m,n)} \subseteq (A)_{(m,n)}$ and by Proposition 2.2.9, the theorem follows. $\square$

Some results that are true for BQ-semigroups need not be true in the case of (m,n)-BQ-semigroups, see for example Theorem 2.2.4. See also the example given below.

Counter Example 3.1.7. Let $X = \{1,2,3,4\}$, $Y = \{1,2\}$. Define a semigroup $S(X,Y) = \{\alpha : X \to X \mid Y\alpha \subseteq Y\}$ with the composition of function and let $A = \left\{\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}\right\}$, $B = \left\{\begin{pmatrix}1&2&3&4\\1&1&4&2\end{pmatrix}\right\}$. Since

$$\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix},$$

$A \subseteq Reg(S(X,Y))$. We see that

$$(B)_{(1,4)} = \left\{\begin{pmatrix}1&2&3&4\\1&1&4&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}\right\} = (B)_{q(1,4)}.$$

However, $(A \cup B)_{(1,4)} \neq (A \cup B)_{q(1,4)}$ since

$\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&1&4&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&2&3&2\end{pmatrix}$ and

$\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix} = \begin{pmatrix}1&2&3&4\\1&1&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&4&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&4&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&1&4&2\end{pmatrix}$,

so $\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix} \in (A \cup B)S(X,Y) \cap S(X,Y)(A \cup B)^4 \subseteq (A \cup B)_{q(1,4)}$, but $\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix} \notin (A \cup B)_{(1,4)}$. Indeed, we put $\alpha = \begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}$ and suppose that $\alpha \in (A \cup B)_{(1,4)}$. Direct computation shows

$$(A \cup B)^2 = \left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&3&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&1\end{pmatrix}\right\}$$
$$(A \cup B)^3 = \left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&4&1\end{pmatrix}\right\}$$
$$(A \cup B)^4 = \left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&4\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&3&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&1\end{pmatrix}\right\}$$
$$(A \cup B)^5 = \left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&4&1\end{pmatrix}\right\}.$$

So $\alpha \notin \bigcup_{i=1}^{1+4}(A \cup B)^i$ and we must have $\alpha \in (A \cup B)S(X,Y)(A \cup B)^4$. That is $\alpha = \lambda\beta\gamma$ for some $\lambda \in A \cup B, \beta \in S(X,Y), \gamma \in (A \cup B)^4$. Since $X\alpha = \{1,2\}$, so γ must be $\begin{pmatrix}1&2&3&4\\1&1&2&1\end{pmatrix}$. If $\lambda = \begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}$, then we have $2 = 3\alpha = 3\lambda\beta\gamma = 1\beta\gamma$. Hence $1\beta = 3$, a contradiction with $\beta \in S(X,Y)$. Similarly, if $\lambda = \begin{pmatrix}1&2&3&4\\1&1&4&2\end{pmatrix}$, then we have $2 = 4\alpha = 4\lambda\beta\gamma = 2\beta\gamma$. Hence $2\beta = 3$, a contradiction with $\beta \in S(X,Y)$. Therefore, $\alpha \notin (A \cup B)_{(1,4)}$. That is $(A \cup B)_{(1,4)} \neq (A \cup B)_{q(1,4)}$.

3.2 Some equivalence relations on (m,n)-BQ-semigroups

For elements a, b in a semigroup S, we write $a\mathcal{B}b$ if and only if $(a)_{(1,1)} = (b)_{(1,1)}$ and write $a\mathcal{Q}b$ if and only if $(a)_{q(1,1)} = (b)_{q(1,1)}$. In [10,18], the authors show that the relations $\mathcal{B}, \mathcal{Q}$ are equivalence relations on S. In [14], Mielke showed that if $S \in BQ$, then $\mathcal{B} = \mathcal{Q}$. In [21], the equivalence relation $\mathcal{B}_m^n$, where m, n are nonnegative integers, was introduced by Tilidetzke. In this thesis, we define the relation $\mathcal{Q}_m^n$ which is more general than the relation $\mathcal{Q}$ and extend some results in [14] to (m,n)-BQ-semigroups as follows.

Definition 3.2.1. Let S be a semigroup and m, n nonnegative integers. For $a, b \in S$, we write $a\mathcal{Q}_m^n b$ if and only if either

(i) $a = b$ or

(ii) $a = b^m u, a = vb^n$ and $b = a^m x, b = ya^n$ for some $u, v, x, y \in S$.

Moreover, we denote the $\mathcal{Q}_m^n$-class containing a by $\mathcal{Q}_m^n(a)$.

Theorem 3.2.2. *The relation $\mathcal{Q}_m^n$ is an equivalence relation. Moreover, $\mathcal{Q}_m^n \subseteq \mathcal{Q}$.*

Proof. The reflexive and symmetric properties are satisfied by definition. Next, we prove that the relation $\mathcal{Q}_m^n$ has transitivity. In case $m = 0$ or $n = 0$, it is easy to see that $\mathcal{Q}_m^n$ has transitivity. Now, we assume that $m, n \neq 0$. Let $a, b, c \in S$ be such that $a\mathcal{Q}_m^n b$ and $b\mathcal{Q}_m^n c$. If $a = b$ or $b = c$, we are done. If $a \neq b$ and $b \neq c$, there are $s, t, u, v, w, x, y, z \in S$ such that $a = b^m s, a = tb^n, b = a^m u, b = va^n, b = c^m w, b = xc^n, c = b^m y$ and $c = zb^n$. Hence

$$a = b^m s = c^m w(c^m w)^{m-1}s, a = tb^n = t(xc^n)^{n-1}xc^n,$$
$$c = b^m y = a^m u(a^m u)^{m-1}y, c = zb^n = z(vb^n)^{n-1}vb^n.$$

Therefore, $a\mathcal{Q}_m^n c$. This proves the relation $\mathcal{Q}_m^n$ is an equivalence relation. It is easy to see that $\mathcal{Q}_m^n \subseteq \mathcal{Q}$. □

Proposition 3.2.3. *Let $a, b \in S$ and m, n nonnegative integers. Then $a\mathcal{Q}_m^n b$ if and only if $(a)_{q(m,n)} = (b)_{q(m,n)}$.*

Proof. Assume that $(a)_{q(m,n)} = (b)_{q(m,n)}$. If $a = b$, then $a\mathcal{Q}_m^n b$. We now suppose that $a \neq b$ and consider the following cases.

Case 1 : $a \in b^m S \cap Sb^n, b \in a^m S \cap Sa^n$. We are done by definition.

Case 2 : $a = b^k, 2 \leq k \leq max\{m, n\}$ and $b \in a^m S \cap Sa^n$. There exist $s, s' \in S$ such that $b = a^m s = s'a^n$. Then

$$a = b^k = (a^m s)^k = (b^{mk} s)^k = b^m b^{m(k-1)} s (b^{mk} s)^{k-1} = b^m u,$$

where $u = b^{m(k-1)} s (b^{mk} s)^{k-1} \in S$. Similarly, we obtain that $a = vb^n$, where $v = (s'b^{nk})^{k-1} s' b^{n(k-1)} \in S$. So $a\mathcal{Q}_m^n b$.

Case 3 : $a \in b^m S \cap Sb^n$ and $b = a^k, 2 \leq k \leq max\{m, n\}$. We can prove in a similar fashion, as above.

Case 4 : $a = b^k, 2 \leq k \leq max\{m, n\}$ and $b = a^l, 2 \leq l \leq max\{m, n\}$. Then

$$a = b^k = a^{lk} = b^{lk^2} = a^{l^2k^2} = b^{l^2k^3} = a^{l^3k^3} = \ldots.$$

We can choose an integer $r > 0$ such that $l^r k^{r+1} > max\{m, n\} + 1$. Hence $a \in b^m S \cap Sb^n$. Similarly, we can show that $b \in a^m S \cap Sa^n$. Therefore $a\mathcal{Q}_m^n b$.

Conversely, we assume that $a\mathcal{Q}_m^n b$. There exist $u, v, x, y \in S$ such that

$$a = b^m u = vb^n, b = a^m x = ya^n.$$

Since $a \in b^m S \cap Sb^n \subseteq (b)_{q(m,n)}$ and $b \in a^m S \cap Sa^n \subseteq (a)_{q(m,n)}$, it follows that

$$(a)_{q(m,n)} \subseteq (b)_{q(m,n)} \subseteq (a)_{q(m,n)}.$$

Therefore, the condition holds. □

Proposition 3.2.4. *For any nonnegative integers m, n, $\mathcal{B}_m^n \subseteq \mathcal{Q}_m^n$.*

Proof. Let $(x, y) \in \mathcal{B}_m^n$. Since $y \in (y)_{(m,n)} = (x)_{(m,n)} \subseteq (x)_{q(m,n)}$, we have $(y)_{q(m,n)} \subseteq (x)_{q(m,n)}$. Similarly, we obtain that $(x)_{q(m,n)} \subseteq (y)_{q(m,n)}$. Thus, $(x)_{q(m,n)} = (y)_{q(m,n)}$, that is, $(x, y) \in \mathcal{Q}_m^n$. □

Lemma 3.2.5. *Let S be a semigroup and m, n be nonnegative integers. For $a \in S$, the following statements are true:*

1. *if $a \in Reg_m^n(S)$, then $\mathcal{B}_m^n(a) = \mathcal{Q}_m^n(a)$;*
2. *if $a \notin Reg_m^n(S)$, then $\mathcal{B}_m^n(a) = \{a\}$.*

Proof. Let $a \in S$.

1. If $a \in Reg_m^n(S)$, then $a = a^m x a^n$ for some $x \in S$. Let $b \in \mathcal{Q}_m^n(a)$ be such that $b \neq a$. In case $m = 0$ or $n = 0$, it is easy to see that $b \in \mathcal{B}_m^n(a)$. Thus, we assume that $m, n > 0$. Hence, there are $u, v, s, t \in S$ such that $a = b^m u = v b^n, b = a^m s = t a^n$. Consider,

$$\begin{aligned} a &= b^m u \\ &= b \cdot b^{m-1} u \\ &= t a^n b^{m-1} u \\ &= t a^{n-1} \cdot a b^{m-1} u \\ &= t a^{n-1} a^m x a^n b^{m-1} u \\ &= t a^n a^{m-1} x a^{n-1} a b^{m-1} u \\ &= a^m s a^{m-1} x a^{n-1} v b^{n-1} b^m u \\ &= b^m u a^{m-1} s a^{m-1} x a^{n-1} v b^{n-1} v b^n \end{aligned}$$

and

$$b = t a^n = t a^n a^{m-1} x a^n = a^m s a^{m-1} x a^n.$$

By definition, we obtain that $b \in \mathcal{B}_m^n(a)$, that is $\mathcal{Q}_m^n(a) \subseteq \mathcal{B}_m^n(a)$. By Proposition 3.2.4, we have $\mathcal{B}_m^n(a) \subseteq \mathcal{Q}_m^n(a)$. Thus, $\mathcal{B}_m^n(a) = \mathcal{Q}_m^n(a)$.

2. If $a \notin Reg_m^n(S)$, we prove by contradiction. Suppose that there is $b \neq a$ and $b \in \mathcal{B}_m^n(a)$. Then

$$a = b^m s b^n = a^m t a^n b^{m-1} s b^{n-1} a^m t a^n \in a^m S a^n$$

for some $s, t \in S$, which is a contradiction. Therefore, $\mathcal{B}_m^n(a) = \{a\}$.

In particular, we can see that if $a \notin Reg(S)$, then $\mathcal{B}(a) = \{a\}$. $\square$

Lemma 3.2.6. *Let S be a semigroup and m, n nonnegative integers. If $Reg(S) = Reg_m^n(S)$, then $\mathcal{B} = \mathcal{B}_m^n$*

Proof. Assume that $Reg(S) = Reg_m^n(S)$. By Proposition 2.3.5, it suffices to show that $\mathcal{B} \subseteq \mathcal{B}_m^n$. Let $(a, b) \in \mathcal{B}$. Then $a = bub, b = ava$ for some $u, v \in S$. If $a \notin Reg_m^n(S) = Reg(S)$ or $b \notin Reg_m^n(S) = Reg(S)$, then $a = b$ by Lemma 3.2.5; hence $(a, b) \in \mathcal{B}_m^n$. If $a, b \in Reg_m^n(S)$, there are $s, t \in S$ such that $a = a^m s a^n$ and $b = b^m t b^n$. Hence $a = b^m t b^n u b^m t b^n$ and $b = a^m s a^n v a^m s a^n$, that is $(a, b) \in \mathcal{B}_m^n$. This means that $\mathcal{B} \subseteq \mathcal{B}_m^n$. $\square$

Theorem 3.2.7. *Let m, n be nonnegative integers. If $S \in BQ_m^n$, then $\mathcal{B}_m^n = \mathcal{Q}_m^n$.*

Proof. Assume that $S \in BQ^n_m$. Let $(x,y) \in \mathcal{Q}^n_m$. Then $(x)_{q(m,n)} = (y)_{q(m,n)}$. By assumption, we obtain

$$(x)_{(m,n)} \subseteq (x)_{q(m,n)} = (y)_{q(m,n)} \subseteq (y)_{(m,n)} \subseteq (y)_{q(m,n)} = (x)_{q(m,n)} \subseteq (x)_{(m,n)}.$$

Hence, $(x,y) \in \mathcal{B}^n_m$ and by proposition 3.2.4, we obtain $\mathcal{B}^n_m = \mathcal{Q}^n_m$. $\square$

The next example show that the converse of Theorem 3.2.7 is not true.

Example 3.2.8. Let X, Y be nonempty sets such that $Y \subseteq X, |X| = 4, |Y| = 3$. The semigroup of transformations with invariant set, denoted by $S(X,Y)$, is defined by $S(X,Y) = \{\alpha : X \to X \mid Y\alpha \subseteq Y\}$. In [8,16], we know that $S(X,Y)$ with composition of functions is a nonregular semigroup. For convenience, let $X = \{1,2,3,4\}, Y = \{1,2,3\}$ and $\beta, \gamma, \lambda_1, \lambda_2 \in S(X,Y)$ given by

$$\beta = \begin{pmatrix}1&2&3&4\\1&1&2&3\end{pmatrix}, \gamma = \begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}, \lambda_1 = \begin{pmatrix}1&2&3&4\\1&3&1&4\end{pmatrix} \text{ and } \lambda_2 = \begin{pmatrix}1&2&3&4\\1&2&1&4\end{pmatrix}.$$

Since $\gamma = \beta^2\lambda_1 = \lambda_2\beta \in \beta^2 S(X,Y) \cap S(X,Y)\beta \subseteq (\beta)_{q(2,1)}$ and $\gamma \neq \beta^i, i = 1,2,3$. If $\gamma \in (\beta)_{(2,1)}$, then $\gamma = \beta^2\lambda_3\beta$ for some $\lambda_3 \in S(X,Y)$. Since $3 = 4\gamma = 4\beta^2\lambda_3\beta = 2\lambda_3\beta$ and $4\beta = 3$, so we must have $2\lambda_3 = 4$, a contradiction with $\lambda_3 \in S(X,Y)$. Thus, $(\beta)_{(2,1)} \neq (\beta)_{q(2,1)}$ and so by Theorem 3.1.3, $S(X,Y) \notin BQ^1_2$. A direct computation shows that $\mathcal{B}^1_2 = \mathcal{Q}^1_2$ as follows,

$$\mathcal{B}^1_2 = \left\{ \left\{\begin{pmatrix}1&2&3&4\\1&1&1&1\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&1&3\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&1&4\end{pmatrix}\right\}, \right.$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&1&2&1\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&2&3\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&2&4\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&1&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&1&3\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&3&2\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&1&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&1&1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&1&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&1&4\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&2&1&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&2&2\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&2&1\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&2&1&3\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&2&1&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&2&4\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&2&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&1&2\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&2&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&1&1\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3&4\\1&2&2&3\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&2&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&1&4\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&3&2&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&3&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&1&3\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&3&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&1&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&2&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&1&2\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&3&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&1&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&1&1\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\1&3&2&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&2&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix}1&2&3&4\\1&3&1&1\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\1&3&1&2\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\1&3&1&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\1&3&1&4\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\1&3&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&1&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\1&3&3&2\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\1&3&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&1&1\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\1&3&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&1&1&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&1&1&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&1&2&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&1&1\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\2&2&1&2\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&1&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&1&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&2&1\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\2&2&2&2\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&2&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&2&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&3&1\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\2&2&3&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&2&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&2&3&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&2&2\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\2&2&3&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&3&2&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&3&2&1\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\2&3&2&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&3&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&3&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&3&2\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\2&3&2&4\end{pmatrix}\begin{pmatrix}1&2&3&4\\3&2&3&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&3&3&1\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&3&3&2\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\2&3&3&3\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\2&3&3&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&1&1&2\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&1&3&1\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&1&3&2\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\3&1&3&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&1&3&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&2&2&1\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&2&2&2\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\3&2&2&3\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&2&2&4\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&2&3&1\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&3&1&2\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\3&3&2&1\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&3&3&1\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&3&3&2\end{pmatrix}\right\},\left\{\begin{pmatrix}1&2&3&4\\3&3&3&3\end{pmatrix}\right\},$$
$$\left\{\begin{pmatrix}1&2&3&4\\3&3&3&4\end{pmatrix}\right\}\Bigg\} = \mathcal{Q}_2^1.$$

Theorem 3.2.9. *Let A be a nonempty subset of a semigroup S and let $\emptyset \neq X \subseteq A$ be such that $|A \cap \mathcal{B}_m^n(x)| = 1$ for all $x \in X$. Then $(A)_{(m,n)} = (A)_{q(m,n)}$ if and only if $(X)_{(m,n)} = (X)_{q(m,n)}$.*

Proof. We first prove that $(X)_{(m,n)} = (A)_{(m,n)}$. Since

$$X = \bigcup_{x\in X}\{x\} \subseteq \bigcup_{x\in X}(x)_{(m,n)} = \bigcup_{a\in A}(a)_{(m,n)} \subseteq (A)_{(m,n)},$$

$(X)_{(m,n)} \subseteq (A)_{(m,n)}$ and since

$$\begin{aligned}(A)_{(m,n)} &= (\bigcup_{a\in A} a)_{(m,n)} \subseteq (\bigcup_{a\in A}(a)_{(m,n)})_{(m,n)}\\ &= (\bigcup_{y\in X}(y)_{(m,n)})_{(m,n)} \subseteq ((X)_{(m,n)})_{(m,n)} = (X)_{(m,n)},\end{aligned}$$

$(X)_{(m,n)} = (A)_{(m,n)}$. Then we asume that $(A)_{(m,n)} = (A)_{q(m,n)}$, which implies that $X \subseteq (A)_{(m,n)} \subseteq (A)_{q(m,n)}$. Hence,

$$(X)_{(m,n)} \subseteq (X)_{q(m,n)} \subseteq (A)_{q(m,n)} = (A)_{(m,n)} = (X)_{(m,n)}.$$

Thus, $(X)_{(m,n)} = (X)_{q(m,n)}$.

Conversely, if $(X)_{(m,n)} = (X)_{q(m,n)}$, then $A \subseteq (A)_{(m,n)} = (X)_{(m,n)} = (X)_{q(m,n)}$, which implies that $(A)_{q(m,n)} \subseteq (X)_{q(m,n)}$. So,

$$(A)_{(m,n)} \subseteq (A)_{q(m,n)} \subseteq (X)_{q(m,n)} = (X)_{(m,n)} = (A)_{(m,n)}$$

and the proof is complete. □

Corollary 3.2.10. *Let S be a semigroup and $a \in S$. Then $(a)_{(m,n)} = (a)_{q(m,n)}$ if and only if for all $C \subseteq \mathcal{B}_m^n(a)$, $(C)_{(m,n)} = (C)_{q(m,n)}$.*

Proof. Assume that $(a)_{(m,n)} = (a)_{q(m,n)}$. Let $C \subseteq \mathcal{B}_m^n(a)$. Since

$$C \subseteq \bigcup_{c \in C} (c)_{(m,n)} = (a)_{(m,n)},$$

$(C)_{(m,n)} \subseteq (a)_{(m,n)}$. Since

$$(a)_{(m,n)} = \bigcup_{c \in C} (c)_{(m,n)} \subseteq (C)_{(m,n)},$$

$(C)_{(m,n)} = (a)_{(m,n)}$. By assumption, we obtain that $(C)_{q(m,n)} \subseteq (C)_{(m,n)}$. Thus $(C)_{(m,n)} = (C)_{q(m,n)}$. The proof for the converse is easy. □

3.3 Some semigroups of transformations which are (m,n)-BQ-semigroups

Let X, Y be nonempty sets such that $Y \subseteq X$ and m, n nonnegative integers. In previous chapter, the concept of full transformation semigroup on X, $T(X)$, and its subsemigroups were introduced. In this section, we characterize these semigroups when they belong to BQ_m^n.

Lemma 3.3.1. *$T(X)$ is an (m,n)-BQ-semigroup.*

Proof. We obtain this Lemma by Theorem 2.4.1 and 3.1.2. □

Theorem 3.3.2. *$T(X,Y)$ is an (m,n)-BQ-semigroup.*

Proof. We prove that $T(X,Y)$ is a left ideal of $T(X)$. Let $\alpha \in T(X,Y)$ and $\beta \in T(X)$. Since $X\beta\alpha \subseteq X\alpha \subseteq Y$, $\beta\alpha \in T(X,Y)$. That is $T(X,Y)$ is a left ideal of $T(X)$. By Theorem 3.1.5 and Theorem 2.4.1, we see that $T(X,Y) = T(X) \cap T(X,Y)$ is an (m,n)-BQ-semigroup. □

By Theorem 2.4.2, we obtain the following theorem.

Theorem 3.3.3. *If one of the following statements holds:*

(*i*) $Y = X$,

(*ii*) $|Y| = 1$,

then the semigroup $S(X,Y) \in BQ_m^n$.

Theorem 3.3.4. *If* $|X| = 3$ *and* $|Y| = 2$, *then* $S(X,Y) \in BQ_m^n$.

Proof. If $m, n = 1$, we are done by Theorem 2.4.4. Assume that $m > 1$ or $n > 1$. For convenience, we let $X = \{1,2,3\}$, $Y = \{1,2\}$. Then

$$\begin{aligned} S := S(X,Y) = \Bigg\{ &\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \\ &\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}, \\ &\begin{pmatrix}1&2&3\\2&2&3\end{pmatrix} \Bigg\}. \end{aligned}$$

In this proof, we want to show that $(A)_{(m,n)} = (A)_{q(m,n)}$ for any nonempty subset A of S. So we divide the proof into four parts as follows.

- **Part I :** To show that $Reg_m^n(S) = Reg(S)$. By Theorem 2.4.3, we have

$$\begin{aligned} Reg(S) = \Bigg\{ &\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \\ &\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&3\end{pmatrix} \Bigg\}. \end{aligned}$$

  It suffices to show that $Reg(S) \subseteq Reg_m^n(S)$. We can see that

$$\begin{aligned} E(S) = \Bigg\{ &\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}, \\ &\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&3\end{pmatrix} \Bigg\}. \end{aligned}$$

  By definition, we can see that every idempotent elements are (m,n)-regular elements. Thus $E(S) \subseteq Reg_m^n(S)$. Since

$$\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}^2 = \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}^2 = \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}^2 = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$$

  are idempotent elements and

$$\begin{aligned} \begin{pmatrix}1&2&3\\2&1&1\end{pmatrix} &= \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix} \\ \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix} &= \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} \end{aligned}$$

$$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix},$$

$$\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}^k=\begin{cases}\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix} & \text{if } k \text{ is odd,}\\ \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix} & \text{if } k \text{ is even,}\end{cases}$$

$$\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}^k=\begin{cases}\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix} & \text{if } k \text{ is odd,}\\ \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} & \text{if } k \text{ is even,}\end{cases}$$

$$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}^k=\begin{cases}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} & \text{if } k \text{ is odd,}\\ \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} & \text{if } k \text{ is even,}\end{cases}$$

which implies that for any m, n,

$$\begin{aligned}\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}&=\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}^m\eta_1\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}^n\\ \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}&=\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}^m\eta_2\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}^n\\ \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}&=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}^m\eta_3\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}^n\end{aligned}$$

where

$$\eta_1=\begin{cases}\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix} & \text{if } m+n \text{ is even,}\\ \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix} & \text{if } m+n \text{ is odd,}\end{cases}$$

$$\eta_2=\begin{cases}\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix} & \text{if } m+n \text{ is even,}\\ \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} & \text{if } m+n \text{ is odd,}\end{cases}$$

$$\eta_3=\begin{cases}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} & \text{if } m+n \text{ is even,}\\ \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} & \text{if } m+n \text{ is odd.}\end{cases}$$

Thus $\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$ are (m,n)-regular elements which implies that

$$Reg(S)=E(S)\cup\left\{\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix},\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\right\}\subseteq Reg_m^n(S).$$

Therefore, $Reg(S)=Reg_m^n(S)$.

- **Part II :** To find $\mathcal{B}_m^n$ on S. By Lemma 3.2.6, we have $\mathcal{B}_m^n = \mathcal{B}$. By Lemma 3.2.5(2), we can compute that

$$\begin{aligned}\mathcal{B}_m^n = \mathcal{B} = \Bigg\{ & \left\{\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}\right\},\\ & \left\{\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}\right\},\\ & \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3\\2&2&1\end{pmatrix}\right\},\\ & \left\{\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}\right\}, \left\{\begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}\right\}\Bigg\},\end{aligned}$$

for any m, n. In the end of this part, we put

$$\begin{aligned}S^* = \Bigg\{ & \begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix},\\ & \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}\Bigg\}\\ & = E(S) \cup \{\alpha, \beta\}\end{aligned}$$

where $\alpha = \begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}, \beta = \begin{pmatrix}1&2&3\\2&2&1\end{pmatrix}$. Note that

$$\alpha^2 = \beta\alpha = \begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \beta^2 = \alpha\beta = \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}.$$

- **Part III :** To claim that for any $A \subseteq E(S)$ and $\emptyset \neq B \subseteq \{\alpha, \beta\}$,

$$(A \cup B)^2 = \begin{cases} C & \text{if } A \subseteq \left\{\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}\right\},\\ A \cup C \cup D & ;\text{otherwise,}\end{cases}$$

where $C \subseteq \left\{\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}\right\}$ such that

$$\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix} \in C \text{ if } Y\eta = \{1\}, \text{ for some } \eta \in A \cup B,$$
$$\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix} \in C \text{ if } Y\eta = \{2\}, \text{ for some } \eta \in A \cup B,$$

and $D \subseteq \{\alpha, \beta\}$ such that

$$D = \begin{cases} B \cup \{\alpha\} & \text{if } \begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&2\end{pmatrix} \in A,\\ B \cup \{\beta\} & \text{if } \begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&2&1\end{pmatrix} \in A,\\ B & ;\text{otherwise.}\end{cases}$$

Indeed, let $A \subseteq E(S)$ and $\emptyset \neq B \subseteq \{\alpha, \beta\}$.

In the case when $A \subseteq \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \right\}$, it is clear that $(A \cup B)^2 = C$.

In another case, we first show that $A \cup D \cup C \subseteq (A \cup B)^2$.

(i) $A \subseteq A^2$ since $A \subseteq E(S)$, i.e. $\theta = \theta^2$ for all $\theta \in A$.

(ii) $D \subseteq A^2 \cup AB \cup BA$ since

$$\begin{aligned}
\alpha &= \alpha \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \\
&= \alpha \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \\
&= \alpha \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \alpha \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \alpha \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\beta &= \beta \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \\
&= \beta \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \\
&= \beta \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \beta \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \beta \\
&= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}.
\end{aligned}$$

(iii) $C \subseteq BA \cup B^2$ by the following:

- If $B = \{\alpha, \beta\}$, then $C = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \right\} \subseteq B^2$
- If $B = \{\alpha\}$ and A contains θ such that $Y\theta = \{2\}$, then

$$C = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \right\}.$$

Since $X\alpha\theta = Y\theta = \{2\}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \in BA$ which implies that

$$C = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \right\} \subseteq BA \cup B^2.$$

· If $B = \{\alpha\}$ and A dose not contain θ such that $Y\theta = \{2\}$, then

$$C = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \right\} \subseteq B^2.$$

· In case $B = \{\beta\}$, we can prove as two above cases that $C \subseteq BA \cup B^2$.

Now, we have $C \subseteq BA \cup B^2$.

From (i), (ii), (iii), we obtain that

$$A \cup C \cup D \subseteq A^2 \cup AB \cup BA \cup B^2 = (A \cup B)^2.$$

Next, we show that $(A \cup B)^2 \subseteq A \cup C \cup D$. It is easy to see that $B^2 \subseteq C$, $A^2 \subseteq A \cup C \cup D$. Since

$$\begin{aligned}
\alpha S^* &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \right\}, \\
S^*\alpha &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \right\}, \\
\beta S^* &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \right\}, \\
S^*\beta &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \right\},
\end{aligned}$$

$AB \cup BA \subseteq C \cup D$. Now, we obtain $(A \cup B)^2 = A \cup C \cup D$.

In the end of this part, we want to show that $(A \cup B)^k = (A \cup B)^2$ for all $k > 1$. Since

$$\begin{aligned}
(A \cup B)^3 &= (A \cup B)(A \cup B)^2 \\
&= \begin{cases} (A \cup B)C = C \\ (A \cup B)(A \cup C \cup D) = (A \cup B)^2 \cup [(D \setminus B) \cup C] \end{cases} \\
&= (A \cup B)^2,
\end{aligned}$$

the result is obtained by induction.

- **Part IV :** Now, we show that $(H)_{(m,n)} = (H)_{q(m,n)}$ for any nonempty subset H of S. According to Theorem 3.2.9, we can reduce to the case of S^* instead of S, and, by Theorem 3.1.6, we obtain that $(F)_{(m,n)} = (F)_{q(m,n)}$ for any $F \subseteq E(S)$. Thus we need show that

$$(A \cup B)_{(m,n)} = (A \cup B)_{q(m,n)}$$

for any $A \subseteq E(S)$ and $\emptyset \neq B \subseteq \{\alpha, \beta\}$. To complete this proof, we consider the following three cases.

Case : $m = 1, n > 1$, we have $(A\cup B)S\cap S(A\cup B)^n = (A\cup B)S\cap S(A\cup B)^2$. Then

$$(A\cup B)S\cap S(A\cup B)^2 = \begin{cases} (A\cup B)S\cap SC \subseteq C = (A\cup B)SC = (A\cup B)S(A\cup B)^2, \\ (A\cup B)S\cap S(A\cup C\cup D). \end{cases}$$

By a part of the proof of Theorem 2.4.4, see [15], we have $(D)_{(1,1)} = (D)_{q(1,1)}$. Since $A\cup C \subseteq Reg(S)$, by Theorem 2.2.4 we obtain that

$$\begin{aligned}
(A\cup B)S\cap S(A\cup B\cup C) \subseteq & (A\cup C\cup D)_{q(1,1)} \\
= & (A\cup C\cup D)_{(1,1)} \\
= & (A\cup C\cup D)\cup(A\cup C\cup D)^2 \\
& \cup(A\cup C\cup D)S(A\cup C\cup D) \\
= & (A\cup B)^2\cup(A\cup B)^4\cup(A\cup C\cup D)S(A\cup C\cup D) \\
= & (A\cup B)^2\cup(A\cup B)S(A\cup C\cup D) \\
& \cup CS(A\cup C\cup D)\cup(D\setminus B)S(A\cup C\cup D) \\
\subseteq & (A\cup B)^2\cup(A\cup B)S(A\cup C\cup D) \\
& \cup(A\cup B)^2\cup A^2S(A\cup C\cup D) \\
\subseteq & (A\cup B)^2\cup(A\cup B)S(A\cup C\cup D) \\
& \cup AS(A\cup C\cup D) \\
\subseteq & (A\cup B)^2\cup(A\cup B)S(A\cup C\cup D) \\
= & (A\cup B)^2\cup(A\cup B)S(A\cup B)^2 \\
\subseteq & (A\cup B)\cup(A\cup B)^2\cup(A\cup B)S(A\cup B)^2 \\
= & (A\cup B)\cup(A\cup B)^2\cup(A\cup B)S(A\cup B)^n \\
\subseteq & (A\cup B)_{(1,n)}.
\end{aligned}$$

Therefore, $(A\cup B)S\cap S(A\cup B)^n \subseteq (A\cup B)_{(1,n)}$.

Case : $m > 1, n = 1$, we have $(A\cup B)^mS\cap S(A\cup B) = (A\cup B)^2S\cap S(A\cup B)$. Then

$$(A\cup B)^2S\cap S(A\cup B) = \begin{cases} CS\cap S(A\cup B), \\ (A\cup C\cup D)S\cap S(A\cup B). \end{cases}$$

If $A \subseteq \left\{\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}\right\}$, then $(A\cup B)^2S\cap S(A\cup B) = CS\cap S(A\cup B) \subseteq CS \subseteq (A\cup B)^2 \subseteq (A\cup B)_{(m,1)}$. If $(A\cup B)^2S\cap S(A\cup B) = (A\cup C\cup D)S\cap S(A\cup B)$, by a part of the proof of Theorem 2.4.4, see [15], we have $(D)_{(1,1)} = (D)_{q(1,1)}$. Since $A \cup C \subseteq Reg(S)$, by Theorem 2.2.4 we obtain that

$$\begin{aligned}
(A \cup C \cup D)S \cap S(A \cup B) \subseteq& (A \cup C \cup D)_{q(1,1)} \\
=& (A \cup C \cup D)_{(1,1)} \\
=& (A \cup C \cup D) \cup (A \cup C \cup D)^2 \\
& \cup (A \cup C \cup D)S(A \cup C \cup D) \\
=& (A \cup B)^2 \cup (A \cup B)^4 \cup (A \cup C \cup D)S(A \cup C \cup D) \\
=& (A \cup B)^2 \cup (A \cup C \cup D)S(A \cup B) \\
& \cup (A \cup C \cup D)S(C \cup (D \setminus B)) \\
=& (A \cup B)^2 \cup (A \cup B)^2 S(A \cup B) \\
& \cup (A \cup B)^2 SC \cup (A \cup B)^2 S(D \setminus B) \\
\subseteq& (A \cup B)^2 \cup (A \cup B)^2 S(A \cup B) \cup C \cup (A \cup B)^2 SA^2 \\
\subseteq& (A \cup B)^2 \cup (A \cup B)^2 S(A \cup B) \cup (A \cup B)^2 \\
& \cup (A \cup B)^2 SA \\
\subseteq& (A \cup B)^2 \cup (A \cup B)^2 S(A \cup B) \\
=& (A \cup B)^2 \cup (A \cup B)^m S(A \cup B) \\
\subseteq& (A \cup B)_{(m,1)}.
\end{aligned}$$

Therefore, $(A \cup B)^m S \cap S(A \cup B) \subseteq (A \cup B)_{(m,1)}$.

Case : $m, n > 1$, we have $(A \cup B)^m S \cap S(A \cup B)^n = (A \cup B)^2 S \cap S(A \cup B)^2$. Then

$$(A \cup B)^2 S \cap S(A \cup B)^2 = \begin{cases} CS \cap SC \subseteq C = CSC = (A \cup B)^2 S(A \cup B)^2, \\ (A \cup C \cup D)S \cap S(A \cup C \cup D). \end{cases}$$

By a part of the proof of Theorem 2.4.4, see [15], we have $(D)_{(1,1)} = (D)_{q(1,1)}$. Since $A \cup C \subseteq Reg(S)$, by Theorem 2.2.4 we obtain that

$$\begin{aligned}
(A \cup C \cup D)S \cap S(A \cup C \cup D) \subseteq& (A \cup C \cup D)_{q(1,1)} \\
=& (A \cup C \cup D)_{(1,1)} \\
=& (A \cup C \cup D) \cup (A \cup C \cup D)^2 \\
& \cup (A \cup C \cup D)S(A \cup C \cup D) \\
=& (A \cup B)^2 \cup (A \cup B)^4 \cup (A \cup B)^2 S(A \cup B)^2
\end{aligned}$$

$$=(A\cup B)^2 \cup (A\cup B)^m S(A\cup B)^n$$
$$\subseteq (A\cup B)_{(m,n)}.$$

Therefore, $(A\cup B)^m S \cap S(A\cup B)^n \subseteq (A\cup B)_{(m,n)}$.

From above cases, since $\displaystyle\bigcup_{i=1}^{max\{m,n\}} (A\cup B)^i \subseteq \bigcup_{i=1}^{m+n} (A\cup B)^i$ for all m,n, $(A\cup B)_{q(m,n)} \subseteq (A\cup B)_{(m,n)}$. By Proposition 2.2.9, we now get $(D)_{(m,n)} = (D)_{q(m,n)}$ for any nonempty subset D of S^*.

Therefore, $S(X,Y) \in BQ_m^n$ and the proof is complete. □

From the above two theorems, we obtain the analogous result on BQ_m^n with BQ, see Theorem 2.4.4, as follows.

Corollary 3.3.5. *If one of the following statements holds*

(i) $Y = X$,

(ii) $|Y| = 1$,

(iii) $|X| \leq 3$,

then $S(X,Y) \in BQ_m^n$.

Since $S(X,Y)$ is a nonregular semigroup, see [8, 16], $S(X,Y)$ need not to be an (m,n)-BQ-semigroup. The following theorems show that $S(X,Y)$ dose not belong to BQ_m^n in some cases.

Theorem 3.3.6. *Let* X, Y *be nonempty sets such that* $|X| > 3, |Y| > 1$ *and* $Y \subsetneq X$. *If* $m = 1$, *then* $S(X,Y) \notin BQ_m^n$.

Proof. If $n = 1$, we are done by Theorem 2.4.4. Suppose that $n > 1$.

- Case $|Y| = 2$. Let $Y = \{a,b\}$. Since $|X| > 3$, so we have $|X \setminus Y| > 1$ and let $c, d \in X \setminus Y$. If $n = 2$, we define $\alpha_1, \beta_1, \gamma_1 \in S(X,Y)$ by

$$\alpha_1 = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & b & c \end{pmatrix},\quad \beta_1 = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & b & a & d \end{pmatrix} \text{ and}$$

$$\gamma_1 = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & b & d & b \end{pmatrix}.$$

Then

$$\begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & b & a \end{pmatrix} = \alpha_1\beta_1 = \gamma_1\alpha_1^2 \in \alpha_1 S(X,Y) \cap S(X,Y)\alpha_1^2 \subseteq (\alpha_1)_{q(1,2)}.$$

Suppose that $\alpha_1\beta_1 \in (\alpha_1)_{(1,2)}$, then $\alpha_1\beta_1 \in \alpha_1 S(X,Y)\alpha_1^2$ because $\alpha_1\beta_1 \notin \{\alpha_1, \alpha_1^2, \alpha_1^3\}$. Then there exists $\eta_1 \in S(X,Y)$ such that $\alpha_1\beta_1 = \alpha_1\eta_1\alpha_1^2$. Hence $b = c\alpha_1\beta_1 = c\alpha_1\eta_1\alpha_1^2 = b\eta_1\alpha_1^2$. Since $\alpha_1^2 = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & a & b \end{pmatrix}$, we have that $b\eta_1 \notin Y$ which is a contradiction with $\eta_1 \in S(X,Y)$. Thus $S(X,Y) \notin BQ_1^2$. If $n > 2$, we define

$$\alpha_2 = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & b & d \end{pmatrix}, \beta_2 = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & b & a & a \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & d & a \end{pmatrix}$$

and let $D = \{\alpha_1, \alpha_2\}$. Since

$$D^n = D^3 = \left\{ \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & a & a \end{pmatrix}, \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & a & b \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & a & c \end{pmatrix}, \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & a & d \end{pmatrix} \right\},$$

for any $n > 2$, we have $\begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & b & a \end{pmatrix} = \alpha_1\beta_2 = \gamma_2\alpha_2\alpha_1\alpha_2 \in DS(X,Y) \cap S(X,Y)D^n \subseteq (D)_{q(1,n)}$. Suppose that $\alpha_1\beta_2 \in (D)_{(1,n)}$, then $\alpha_1\beta_2 \in DS(X,Y)D^n$ because $\alpha_1\beta_2 \notin \bigcup_{i=1}^{1+n} D^i$; there exist $\eta_2 \in S(X,Y)$, $\alpha \in D$ and $\lambda^* \in D^n$ such that $\alpha_1\beta_2 = \alpha\eta_2\lambda^*$. Since $c\alpha_1 = c\alpha_2 = b$, so $b = c\alpha_1\beta_2 = c\alpha\eta_2\lambda^* = b\eta_2\lambda^*$. Thus we must have $\lambda^* = \begin{pmatrix} a & b & c & X\setminus\{a,b,c\} \\ a & a & a & b \end{pmatrix} \in D^3$, which implies that $b\eta_2 \notin Y$, a contradiction to $\eta_2 \in S(X,Y)$ Thus $S(X,Y) \notin BQ_1^n$.

- Case $|Y| > 2$. Since $X \setminus Y \neq \emptyset$, we can assume that $a, b, c \in Y$ and $d \in X \setminus Y$. Now, we define $\alpha_1, \alpha_2, \beta_1, \beta_2$ by

$$\alpha_1 = \begin{pmatrix} a & b & Y\setminus\{a,b\} & X\setminus Y \\ a & a & c & d \end{pmatrix}, \alpha_2 = \begin{pmatrix} a & Y\setminus\{a\} & X\setminus Y \\ a & b & c \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} a & b & Y\setminus\{a,b\} & x \\ b & a & c & x \end{pmatrix}_{x\in X\setminus Y}, \beta_2 = \begin{pmatrix} a & Y\setminus\{a\} & x \\ c & a & x \end{pmatrix}_{x\in X\setminus Y}.$$

It easy to see that $\alpha_2\alpha_1 = \alpha_1\alpha_2\alpha_1$, $\alpha_1^2 = \alpha_1$ and $\alpha_2\alpha_1\alpha_2 = \alpha_2^3 = \alpha_2^2$. Define $D = \{\alpha_1, \alpha_2\}$. Then we have $D^2 = \{\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_1, \alpha_2^2\}$ and

$$D^3 = \{\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_1, \alpha_2^2, \alpha_2^2\alpha_1\} = D^4 = \ldots = D^n.$$

Hence $\alpha_2\beta_1 = \beta_2\alpha_1\alpha_2 \in DS(X,Y) \cap S(X,Y)D^n \subseteq (D)_{q(1,n)}$. Suppose that $\alpha_2\beta_1 \in (D)_{(1,n)}$. Since $\alpha_2\beta_1 \notin \bigcup_{i=1}^{n+1} D^i$, $\alpha_2\beta_1 \in DS(X,Y)D^n = DS(X,Y)D^3$ that is $\alpha_2\beta_1 = \alpha\eta\lambda$ for some $\alpha \in D$, $\eta \in S(X,Y)$, $\lambda \in D^3$. Since

$$\alpha_2\beta_1 = \begin{pmatrix} a & Y\setminus\{a\} & X\setminus Y \\ b & a & c \end{pmatrix}$$

and $(X\alpha_2)\beta_1 \neq X\alpha^*$ for all $\alpha^* \in D^3 \setminus \{\alpha_1\alpha_2\}$, we must have $\lambda = \alpha_1\alpha_2 = \begin{pmatrix} a & b & Y\setminus\{a,b\} & X\setminus Y \\ a & a & b & c \end{pmatrix}$. If $\alpha = \alpha_1$, then $b = a\alpha_2\beta_1 = a\alpha_1\eta\lambda = b\alpha_1\eta\lambda = b\alpha_2\beta_1 = a$ which is a contradiction. If $\alpha = \alpha_2$, then $c = d\alpha_2\beta_1 = d\alpha_2\eta\alpha_1\alpha_2 = c\eta\alpha_1\alpha_2$ implies that $c\eta \in X \setminus Y$ which is a contradiction to $\eta \in S(X,Y)$. Thus $\alpha_2\beta_1 \notin (D)_{(1,n)}$, that is $(D)_{(1,n)} \neq (D)_{q(1,n)}$. Therefore, $S(X,Y) \notin BQ_1^n$

□

Theorem 3.3.7. *Let X, Y be sets such that $|X| = 4, |Y| = 2$ and $Y \subsetneq X$. If one of the following statements holds*

(*i*) $n = 1$,

(*ii*) $m = 2, n = 2$,

then $S(X,Y) \notin BQ_m^n$.

Proof. Let $X = \{a,b,c,d\}$ and $Y = \{a,b\}$. Define

$$\mu_1 = \begin{pmatrix} a & b & c & d \\ a & a & b & c \end{pmatrix}, \mu_2 = \begin{pmatrix} a & b & c & d \\ a & a & b & d \end{pmatrix}$$
$$\gamma_1 = \begin{pmatrix} a & b & c & d \\ a & b & b & c \end{pmatrix}, \gamma_2 = \begin{pmatrix} a & b & c & d \\ a & b & b & d \end{pmatrix}$$
$$\rho = \begin{pmatrix} a & b & c & d \\ b & b & d & a \end{pmatrix}, \beta = \begin{pmatrix} a & b & c & d \\ a & a & c & b \end{pmatrix}.$$

(*i*) Assume that $n = 1$. If $m = 1$, we are done by Theorem 2.4.4. Let $m > 1$ and $D = \{\mu_1, \gamma_2, \rho\}$. Then we have

$$D^2 = \left\{ \begin{pmatrix} a & b & c & d \\ a & a & a & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & b & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & a & c \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} a & b & c & d \\ a & b & b & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & c & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & d & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & a & b \end{pmatrix} \right\},$$
$$D^3 = D^2 \cup \left\{ \begin{pmatrix} a & b & c & d \\ a & a & a & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & b & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & d & b \end{pmatrix} \right\}.$$

It's easy to see that $D^2 \subseteq D^3 = D^4 = D^5 = \ldots$. Since

$$\beta = \begin{pmatrix} a & b & c & d \\ b & b & d & a \end{pmatrix}\begin{pmatrix} a & b & c & d \\ b & a & c & c \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ a & a & d & c \end{pmatrix}\begin{pmatrix} a & b & c & d \\ a & a & b & c \end{pmatrix},$$

$\beta \in D^2S(X,Y) \cap S(X,Y)D \subseteq (D)_{q(2,1)} \subseteq (D)_{q(3,1)} \subseteq (D)_{q(m,1)}$. Suppose that $S(X,Y) \in BQ_m^1$. Then $(D)_{q(m,1)} = (D)_{(m,1)}$. Since $\beta \notin \bigcup_{i\in\mathbb{N}} D^i$, $\beta \in D^mS(X,Y)D$, that is $\beta = \lambda\eta\alpha$ for some $\lambda \in D^m, \eta \in S(X,Y), \alpha \in D$. Since $c \in X\beta$, we must have $\alpha = \begin{pmatrix} a & b & c & d \\ a & a & b & c \end{pmatrix}$. If $\lambda = \begin{pmatrix} a & b & c & d \\ a & a & c & a \end{pmatrix}$ or $\begin{pmatrix} a & b & c & d \\ b & b & d & a \end{pmatrix}$ or $\begin{pmatrix} a & b & c & d \\ b & b & d & b \end{pmatrix}$, then $d\lambda \in Y$. Then we get a contradiction from $b = d\beta = d\lambda\eta\alpha = a$. In the other hand, we have $c = c\beta = c\lambda\eta\alpha = a$ which is a contradiction. Therefore, $S(X,Y) \notin BQ_m^1$.

(ii) Assume that $m = n = 2$. Let $D = \{\mu_2, \gamma_1, \rho\}$. Then we have

$$\begin{aligned}
D^2 =& \Bigg\{ \begin{pmatrix} a & b & c & d \\ a & a & a & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & a & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & b & b & b \end{pmatrix}, \\
& \begin{pmatrix} a & b & c & d \\ b & b & b & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & d & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & a & b \end{pmatrix} \Bigg\}, \\
D^3 =& D^2 \cup \Bigg\{ \begin{pmatrix} a & b & c & d \\ a & a & a & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & b & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & a & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & b \end{pmatrix}, \\
& \begin{pmatrix} a & b & c & d \\ b & b & b & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & c & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ a & a & b & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & d & b \end{pmatrix} \Bigg\}, \\
D^4 =& D^3 \cup \Bigg\{ \begin{pmatrix} a & b & c & d \\ b & b & c & b \end{pmatrix} \Bigg\}.
\end{aligned}$$

Suppose that $S(X,Y) \in BQ_2^2$. Since

$$\beta = \begin{pmatrix} a & b & c & d \\ b & b & c & a \end{pmatrix}\begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}\begin{pmatrix} a & b & c & d \\ a & a & b & c \end{pmatrix},$$

$\beta \in D^2S(X,Y) \cap S(X,Y)D^2 \subseteq (D)_{q(2,2)} = (D)_{(2,2)}$. Since $\beta \notin \bigcup_{i=1}^{4} D^i$, there are $\lambda, \alpha \in D^2, \eta \in S(X,Y)$ such that $\beta = \lambda\eta\alpha$. Since $c \in X\beta$ and $a = Y\beta = Y\lambda\eta\alpha$, we must have $\alpha = \begin{pmatrix} a & b & c & d \\ a & a & b & c \end{pmatrix}$. If $c\lambda \in Y$, then $c = c\beta = c\lambda\eta\alpha = a$, a contradiction. If $c\lambda \notin Y$, then $d\lambda = a$ for all $\lambda \in D^2$. Consider $b = d\beta = d\lambda\eta\alpha = a\eta\alpha = a$ which is a contradiction. Thus $S(X,Y) \notin BQ_2^2$.

□

Theorem 3.3.8. *Let X, Y be sets such that $|X| = 4, |Y| = 3$ and $Y \subsetneq X$. If one of the following statements holds*

(i) $n = 1$,

(ii) $m = 2, n = 2$,

then $S(X,Y) \notin BQ_m^n$.

Proof. Let $X = \{a, b, c, d\}$ and $Y = \{a, b, c\}$. Suppose that $S(X,Y) \in BQ_m^n$.

(i) Assume that $n = 1$. If $m = 1$, we are done by Theorem 2.4.4. Let $m > 1$ and $D = \Bigg\{ \begin{pmatrix} a & b & c & d \\ c & c & c & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & b & c & a \end{pmatrix} \Bigg\}$. Then we can compute that

$$D^2 = \Bigg\{ \begin{pmatrix} a & b & c & d \\ c & c & c & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & c & c & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & b & c & c \end{pmatrix} \Bigg\}.$$

Given $\beta = \begin{pmatrix} a & b & c & d \\ c & c & c & a \end{pmatrix} \in S(X,Y)$, since $D^2 = D^3 = D^4 = \ldots$ and

$$\beta = \begin{pmatrix} a & b & c & d \\ c & c & c & b \end{pmatrix}\begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ a & c & c & d \end{pmatrix}\begin{pmatrix} a & b & c & d \\ c & b & c & a \end{pmatrix},$$

$\beta \in D^2S(X,Y)\cap S(X,Y)D \subseteq (D)_{q(2,1)} = (D)_{q(m,1)} = (D)_{(m,1)}$. Since $\beta \notin \bigcup_{i=1}^{m+1} D^i$, $\beta \in D^mS(X,Y)D$; that is $\beta = \lambda\eta\alpha$ for some $\lambda \in D^2, \eta \in S(X,Y), \alpha \in D$. Since $a \in X\beta$, we must have $\alpha = \begin{pmatrix} a & b & c & d \\ c & b & c & a \end{pmatrix}$. Since $a = d\beta = d\lambda\eta\alpha$ and $d\lambda \in Y$ for all $\lambda \in D^2$, we obtain that $d\lambda\eta = d$ which is a contradiction to $\eta \in S(X,Y)$.

(ii) Assume that $m = n = 2$. Let $D = \left\{ \begin{pmatrix} a & b & c & d \\ c & b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & b & b & d \end{pmatrix} \right\}$. It is easy to compute that

$$D^2 = \left\{ \begin{pmatrix} a & b & c & d \\ c & b & c & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & b & b & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & d \end{pmatrix} \right\},$$
$$D^3 = D^2 \cup \left\{ \begin{pmatrix} a & b & c & d \\ b & b & b & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & b & b & c \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & b & b & a \end{pmatrix} \right\}$$

and $D^2 \subseteq D^3 = D^4 = D^5 = D^6 = \ldots$. Define $\beta = \begin{pmatrix} a & b & c & d \\ b & c & c & a \end{pmatrix}$. Since

$$\beta = \begin{pmatrix} a & b & c & d \\ c & b & b & a \end{pmatrix}\begin{pmatrix} a & b & c & d \\ a & c & b & d \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ b & a & a & d \end{pmatrix}\begin{pmatrix} a & b & c & d \\ c & b & b & a \end{pmatrix},$$

$\beta \in D^2S(X,Y) \cap S(X,Y)D^2 \subseteq (D)_{q(2,2)} = (D)_{(2,2)}$. Since $\beta \notin D \cup D^2 \cup D^3 = D \cup D^2 \cup D^3 \cup \ldots \cup D^{m+1}$, there are $\lambda, \alpha \in D^2, \eta \in S(X,Y)$ such that $\beta = \lambda\eta\alpha$. Since $a \in X\beta$, we must have $\alpha = \begin{pmatrix} a & b & c & d \\ c & b & b & a \end{pmatrix}$. If $\lambda = \begin{pmatrix} a & b & c & d \\ b & b & b & d \end{pmatrix}$, then $b = a\beta = a\lambda\eta\alpha = b\eta\alpha = b\lambda\eta\alpha = b\beta = c$ which is a contradiction. If $\lambda \neq \begin{pmatrix} a & b & c & d \\ b & b & b & d \end{pmatrix}$, then $d\lambda \in Y$ for all λ and implies that $d\lambda\eta \in Y$. Since $a = d\beta = d\lambda\eta\alpha$, $d\lambda\eta = d$. Hence $d\lambda\eta \notin Y$, a contradiction.

Therefore, $S(X,Y) \notin BQ_m^n$. $\square$

Theorem 3.3.9. *Let X, Y be sets. If $|X| > 4, |Y| > 1$ and $Y \subsetneq X$, then $S(X,Y) \notin BQ_m^n$.*

Proof. Let $m, n \in \mathbb{N}$. The case where $m = 1$ and $n \in \mathbb{N}$ was proved in Theorem 3.3.6. We assume that $m > 1$. Since $Y \subsetneq X$, so $|X \setminus Y| \geq 1$.

- If $|X \setminus Y| = 1$, then $|Y| > 3$ because $|X| > 4$. Let $a, b, c, d \in Y$ and $X \setminus Y = \{e\}$. If $n = 1$, we define

$$\alpha_1 = \begin{pmatrix} Y & e \\ d & b \end{pmatrix}, \alpha_2 = \begin{pmatrix} \{a,b\} & c & Y \setminus \{a,b,c\} & e \\ a & c & d & b \end{pmatrix},$$

and let $D = \{\alpha_1, \alpha_2\}$. Then we can easily compute that

$$D^k = D^2 = \left\{ \begin{pmatrix} X \\ d \end{pmatrix}, \begin{pmatrix} Y & e \\ d & a \end{pmatrix}, \begin{pmatrix} \{a,b,e\} & c & Y \setminus \{a,b,c\} \\ a & c & d \end{pmatrix} \right\},$$

for any $k > 2$. Since

$$\begin{pmatrix} Y & e \\ a & b \end{pmatrix} = \begin{pmatrix} Y & e \\ d & a \end{pmatrix}\begin{pmatrix} d & Y \setminus \{d\} & e \\ a & b & a \end{pmatrix}$$

$$= \begin{pmatrix} Y & e \\ a & e \end{pmatrix} \begin{pmatrix} \{a,b\} & c & Y\setminus\{a,b,c\} & e \\ a & c & d & b \end{pmatrix}$$
$$\in D^2S(X,Y)\cap S(X,Y)D = D^mS(X,Y)\cap S(X,Y)D,$$

we have $\begin{pmatrix} Y & e \\ a & b \end{pmatrix} \in (D)_{q(m,1)}$. Suppose that $\begin{pmatrix} Y & e \\ a & b \end{pmatrix} \in (D)_{(m,1)}$. Since $\begin{pmatrix} Y & e \\ a & b \end{pmatrix} \notin \bigcup\limits_{i=1}^{m+1} D^i$, there are $\lambda \in D^m, \eta \in S(X,Y), \alpha \in D$ such that $\begin{pmatrix} Y & e \\ a & b \end{pmatrix} = \lambda\eta\alpha$. Since $a \in X\begin{pmatrix} Y & e \\ a & b \end{pmatrix}$, $\alpha = \alpha_2$. If $\lambda = \begin{pmatrix} X \\ d \end{pmatrix}$, then $\left|X\begin{pmatrix} Y & e \\ a & b \end{pmatrix}\right| > |((X\lambda)\eta)\alpha|$, which is impossible. If $\lambda \neq \begin{pmatrix} X \\ d \end{pmatrix}$, then $b = e\lambda\eta\alpha_2 = a\eta\alpha_2$ implies $a\eta = e$ which is a contradiction. Thus $(D)_{q(m,1)} \neq (D)_{(m,1)}$, that is $S(X,Y) \notin BQ_m^1$. In case $n > 1$, we define

$$\alpha_3 = \begin{pmatrix} \{a,b,c\} & Y\setminus\{a,b,c\} & e \\ b & a & c \end{pmatrix}, \alpha_4 = \begin{pmatrix} a & b & Y\setminus\{a,b\} & e \\ a & b & d & e \end{pmatrix},$$

and put $E = \{\alpha_3, \alpha_4\}$. A direct computation shows that

$$E^2 = \left\{ \begin{pmatrix} X \\ b \end{pmatrix}, \begin{pmatrix} \{a,b,c\} & Y\setminus\{a,b,c\} & e \\ b & a & d \end{pmatrix}, \begin{pmatrix} \{a,b\} & Y\setminus\{a,b\} & e \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & b & Y\setminus\{a,b\} & e \\ a & b & d & e \end{pmatrix} \right\},$$
$$E^3 = \left\{ \begin{pmatrix} X \\ b \end{pmatrix}, \begin{pmatrix} Y & e \\ b & a \end{pmatrix}, \begin{pmatrix} \{a,b,c\} & Y\setminus\{a,b,c\} & e \\ b & a & d \end{pmatrix}, \begin{pmatrix} \{a,b\} & Y\setminus\{a,b\} & e \\ b & a & d \end{pmatrix}, \begin{pmatrix} \{a,b\} & Y\setminus\{a,b\} & e \\ b & a & c \end{pmatrix}, \begin{pmatrix} a & b & Y\setminus\{a,b\} & e \\ a & b & d & e \end{pmatrix} \right\}.$$

It is easy to see that $E^2 \subseteq E^3 = E^4 = E^5 = \ldots$ and we define

$$\beta = \begin{pmatrix} \{a,b,c\} & Y\setminus\{a,b,c\} & e \\ a & b & c \end{pmatrix}.$$

Since

$$\begin{aligned} \beta &= \begin{pmatrix} \{a,b,c\} & Y\setminus\{a,b,c\} & e \\ b & a & d \end{pmatrix} \begin{pmatrix} a & \{b,c\} & Y\setminus\{a,b,c\} & e \\ b & a & c & e \end{pmatrix} \\ &= \begin{pmatrix} \{a,b\} & c & Y\setminus\{a,b,c\} & e \\ c & d & a & e \end{pmatrix} \begin{pmatrix} \{a,b\} & Y\setminus\{a,b\} & e \\ b & a & c \end{pmatrix} \\ &\in E^2S(X,Y)\cap S(X,Y)E^2 \cap E^3S(X,Y)\cap S(X,Y)E^3, \end{aligned}$$

we obtain that $\beta \in \bigcap\limits_{i,j\in\{2,3\}} (E)_{q(i,j)} \subseteq (E)_{q(m,n)}$. Suppose that $S(X,Y) \in BQ_m^n$. Then $(E)_{q(m,n)} = (E)_{(m,n)}$. Since $\beta \notin \bigcup\limits_{i=1}^{m+n} E^i$, $\beta = \lambda\eta\alpha$ for some $\lambda \in E^m, \eta \in S(X,Y), \alpha \in E^n$. Since $c \in X\beta$, we must have $\alpha = \begin{pmatrix} \{a,b\} & Y\setminus\{a,b\} & e \\ b & a & c \end{pmatrix}$. Since $|X\beta| = 3$, λ can be one of the possible cases:

If $\lambda = \begin{pmatrix} \{a,b,c\} & Y \setminus \{a,b,c\} & e \\ b & a & d \end{pmatrix}$, then $c = e\beta = e\lambda\eta\alpha = d\eta\alpha$ implies that $d\eta = e$ which is a conradiction.

If $\lambda = \begin{pmatrix} \{a,b\} & Y \setminus \{a,b\} & e \\ b & a & c \end{pmatrix}$, then $c = e\beta = e\lambda\eta\alpha = c\eta\alpha$ implies that $c\eta = e$ which is a contradiction.

If $\lambda = \begin{pmatrix} a & b & Y \setminus \{a,b\} & e \\ a & b & d & e \end{pmatrix}$, then $a = c\beta = c\lambda\eta\alpha = d\lambda\eta\alpha = d\beta = b$ which is a contradiction.

If $\lambda = \begin{pmatrix} \{a,b\} & Y \setminus \{a,b\} & e \\ b & a & d \end{pmatrix}$, then $a = c\beta = c\lambda\eta\alpha = d\lambda\eta\alpha = d\beta = b$ which is a contradiction.

Hence $\beta \notin (E)_{(m,n)}$, that is $(E)_{q(m,n)} \neq (E)_{(m,n)}$. Therefore, $S(X,Y) \notin BQ_m^n$.

- If $|X \setminus Y| = 2$. Since $|X| > 4$, we have $|Y| > 2$. Let $a,b,c \in Y$ and $X \setminus Y = \{d,e\}$. In this case, we define

$$D = \left\{ \begin{pmatrix} b & Y \setminus \{b\} & \{d,e\} \\ b & a & d \end{pmatrix}, \begin{pmatrix} Y & d & e \\ a & c & b \end{pmatrix} \right\}$$

and compute that

$$D^2 = \left\{ \begin{pmatrix} b & Y \setminus \{b\} & \{d,e\} \\ b & a & d \end{pmatrix}, \begin{pmatrix} Y & \{d,e\} \\ a & c \end{pmatrix}, \begin{pmatrix} Y \cup \{d\} & e \\ a & b \end{pmatrix}, \begin{pmatrix} X \\ a \end{pmatrix} \right\}.$$

It is clear that $D^2 = D^3 = D^4 = \ldots$. So we let $\beta = \begin{pmatrix} Y \cup \{d\} & e \\ a & c \end{pmatrix}$ and since

$$\begin{aligned} \beta &= \begin{pmatrix} Y \cup \{d\} & e \\ a & d \end{pmatrix} \begin{pmatrix} Y & d & e \\ a & c & b \end{pmatrix} \\ &= \begin{pmatrix} Y \cup \{d\} & e \\ a & b \end{pmatrix} \begin{pmatrix} a & b & x \\ a & c & x \end{pmatrix}_{x \in X \setminus \{a,b\}} \\ &= \begin{pmatrix} Y \cup \{d\} & e \\ a & e \end{pmatrix} \begin{pmatrix} Y & \{d,e\} \\ a & c \end{pmatrix} \\ &\in S(X,Y)D \cap D^2 S(X,Y) \cap S(X,Y)D^2, \end{aligned}$$

we get that $\beta \in (D)_{q(m,n)}$ for $n \geq 1$. If $S(X,Y) \in BQ_m^n$, then $\beta \in (D)_{(m,n)}$. Since $\beta \notin \bigcup_{i \in \mathbb{N}} D^i$, β must belong to $(D)_{(m,n)}$; that is $\beta = \lambda\eta\alpha$ for some $\lambda \in D^m = D^2, \eta \in S(X,Y), \alpha \in D^n = D \cup D^2$. From $c \in X\beta$, if $n = 1$, we must have $\alpha = \begin{pmatrix} Y & d & e \\ a & c & b \end{pmatrix}$. Since $|X\beta| = 2$, $\lambda \neq \begin{pmatrix} X \\ a \end{pmatrix}$. If $\lambda = \begin{pmatrix} b & Y \setminus \{b\} & \{d,e\} \\ b & a & d \end{pmatrix}$ or $\begin{pmatrix} Y & \{d,e\} \\ a & c \end{pmatrix}$, then $a = d\beta = d\lambda\eta\alpha = e\lambda\eta\alpha = e\beta = c$ which is a contradiction. If $\lambda = \begin{pmatrix} Y \cup \{d\} & e \\ a & b \end{pmatrix}$, then $c = e\beta = e\lambda\eta\alpha = b\eta\alpha = a$ because $Y\alpha = a$, a contradiction. Thus $S(X,Y) \notin BQ_m^1$. Now, we assume that $n > 1$. Then we also have $\alpha = \begin{pmatrix} Y & \{d,e\} \\ a & c \end{pmatrix}$. With the same reason in case $n = 1$, we get a contradiction. Therefore, $S(X,Y) \notin BQ_m^n$.

- If $|X \setminus Y| > 2$, then there are $c, d, e \in X \setminus Y$. Since $|Y| > 1$, we can let $a, b \in Y$. In case $m = 2$, we define

$$\alpha_1 = \begin{pmatrix} c & d & X \setminus \{c,d\} \\ e & d & b \end{pmatrix}, \alpha_2 = \begin{pmatrix} Y & \{c,d\} & x \\ b & e & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}$$

and let $D = \{\alpha_1, \alpha_2\}$. It is easy to compute the following sets:

$$D^2 = \left\{ \begin{pmatrix} d & X \setminus \{d\} \\ d & b \end{pmatrix}, \begin{pmatrix} c & d & X \setminus \{c,d\} \\ a & e & b \end{pmatrix}, \begin{pmatrix} X \\ b \end{pmatrix}, \begin{pmatrix} \{c,d\} & X \setminus \{c,d\} \\ a & b \end{pmatrix} \right\},$$

$$D^3 = \left\{ \begin{pmatrix} d & X \setminus \{d\} \\ d & b \end{pmatrix}, \begin{pmatrix} d & X \setminus \{d\} \\ e & b \end{pmatrix}, \begin{pmatrix} X \\ b \end{pmatrix}, \begin{pmatrix} d & X \setminus \{d\} \\ a & b \end{pmatrix} \right\}$$

and we can see that $D^3 = D^4 = D^5 = \ldots$. Given $\beta = \begin{pmatrix} c & X \setminus \{c\} \\ a & b \end{pmatrix}$, we see that $\beta \notin \bigcup\limits_{i \in \mathbb{N}} D^i$. Since

$$\begin{aligned} \beta &= \begin{pmatrix} c & X \setminus \{c\} \\ e & b \end{pmatrix} \alpha_2 \\ &= \begin{pmatrix} c & d & X \setminus \{c,d\} \\ a & e & b \end{pmatrix} \begin{pmatrix} a & X \setminus \{a\} \\ a & b \end{pmatrix} \\ &= \begin{pmatrix} c & X \setminus \{c\} \\ c & b \end{pmatrix} \begin{pmatrix} \{c,d\} & X \setminus \{c,d\} \\ a & b \end{pmatrix} \\ &= \begin{pmatrix} c & X \setminus \{c\} \\ d & b \end{pmatrix} \begin{pmatrix} d & X \setminus \{d\} \\ a & b \end{pmatrix} \\ &\in S(X,Y)D \cap D^2 S(X,Y) \cap S(X,Y)D^2 \cap S(X,Y)D^3, \end{aligned}$$

we have $\beta \in (D)_{q(2,1)} \cap (D)_{q(2,2)} \cap (D)_{q(2,3)}$. Suppose that $S(X,Y) \in BQ_2^n$ for some $n \in \mathbb{N}$. From $D^3 = D^4 = D^5 = \ldots$, we can consider n in 3 cases as follows.

Case $n = 1$, we have $\beta \in (D)_{(2,1)}$. Then $\beta = \lambda\eta\alpha$ for some $\lambda \in D^2, \eta \in S(X,Y), \alpha \in D$. Since $c\lambda \in Y$ for all $\lambda \in D^2$, $c\lambda\eta \in Y$ and we must have $\alpha = \alpha_2$ because $a \in X\beta$. Since $a = c\beta = c\lambda\eta\alpha_2$, $c\lambda\eta \notin Y$ which is a contradiction.

Case $n = 2$, we have $\beta \in (D)_{(2,2)}$. Then there are $\lambda, \alpha \in D^2, \eta \in S(X,Y)$ such that $\beta = \lambda\eta\alpha$. Since $a \in X\beta$, we must have

$$\alpha = \begin{pmatrix} c & d & X \setminus \{c,d\} \\ a & e & b \end{pmatrix} \text{ or } \begin{pmatrix} \{c,d\} & X \setminus \{c,d\} \\ a & b \end{pmatrix}.$$

Since $c\lambda \in Y$ for all $\lambda \in D^2$, $c\lambda\eta \in Y$. Hence $b = c\lambda\eta\alpha = c\beta = a$ which is a contradiction.

Case $n \geq 3$, we have $\beta \in (D)_{(2,n)}$. Similarly, we also obtain that $\beta = \lambda\eta\alpha$ for some $\lambda \in D^2, \eta \in S(X,Y), \alpha \in D^n$. Since $a \in X\beta$, we must have $\alpha = \begin{pmatrix} d & X \setminus \{d\} \\ a & b \end{pmatrix}$. Since $c\lambda \in Y$ for all $\lambda \in D^2$, $c\lambda\eta \in Y$ implies that $a = c\beta = c\lambda\eta\alpha = b$ which is a contradiction.

Therefore, $S(X,Y) \notin BQ_2^n$. Now, we assume that $m > 2$. If $n = 1$ or $n = 2$, we define

$$D = \left\{ \begin{pmatrix} Y & c & d & x \\ b & e & d & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}, \begin{pmatrix} a & Y \setminus \{a\} & c & d & x \\ a & b & e & b & x \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \right\}.$$

We see that

$$\begin{aligned} D^2 =& \left\{ \begin{pmatrix} c & d & X \setminus \{c,d\} \\ a & d & b \end{pmatrix}, \begin{pmatrix} Y & c & d & x \\ b & e & b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}, \begin{pmatrix} Y \cup \{d\} & x \\ b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{d\})} \right. \\ & \left. \begin{pmatrix} a & Y \setminus \{a\} & c & d & x \\ a & b & e & b & x \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \right\}, \\ D^3 =& \left\{ \begin{pmatrix} d & X \setminus \{d\} \\ d & b \end{pmatrix}, \begin{pmatrix} c & X \setminus \{c\} \\ a & b \end{pmatrix}, \begin{pmatrix} Y & c & d & x \\ b & e & b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}, \begin{pmatrix} X \\ b \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} Y \cup \{d\} & x \\ b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{d\})}, \begin{pmatrix} a & Y \setminus \{a\} & c & d & x \\ a & b & e & b & x \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \right\} \end{aligned}$$

and $D^3 = D^4 = D^5 = \ldots$. Let $\beta = \begin{pmatrix} Y & c & d & x \\ b & d & b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}$. Then we have

$$\begin{aligned} \beta =& \begin{pmatrix} Y & c & d & x \\ b & e & b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \begin{pmatrix} a & Y \setminus \{a\} & x \\ a & b & d \end{pmatrix}_{x \in X \setminus Y} \\ =& \begin{pmatrix} Y \cup \{d\} & c & x \\ b & d & x \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \begin{pmatrix} Y & c & d & x \\ b & e & d & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \\ =& \begin{pmatrix} Y & c & d & x \\ b & d & b & c \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \begin{pmatrix} c & d & X \setminus \{c,d\} \\ a & d & b \end{pmatrix} \\ \in& D^m S(X,Y) D \cap S(X,Y) D \cap S(X,Y) D^2, \end{aligned}$$

hence $\beta \in (D)_{q(m,2)} \cap (D)_{q(m,1)}$. We suppose that $S(X,Y) \in BQ_m^n$, $n \in \{1,2\}$. Since $\beta \notin \bigcup\limits_{i \in \mathbb{N}} D^i$, we obtain that $\beta = \lambda\eta\alpha$ for some $\lambda \in D^m, \eta \in S(X,Y), \alpha \in D \cup D^2$. Since $d \in X\beta$, α can be $\begin{pmatrix} Y & c & d & x \\ b & e & d & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}$ or $\begin{pmatrix} c & d & X \setminus \{c,d\} \\ a & d & b \end{pmatrix}$. If $\lambda = \begin{pmatrix} Y \cup \{d\} & x \\ b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{d\})}$, then we obtain by $c\lambda = e\lambda$ that $d = c\beta = c\lambda\eta\alpha = e\lambda\eta\alpha = e\beta = a$ which is a contradiction. If $\lambda \neq \begin{pmatrix} Y \cup \{d\} & x \\ b & a \end{pmatrix}_{x \in X \setminus (Y \cup \{d\})}$, by $e\lambda \in Y$ we also get that $e\lambda\eta \in Y$. Since $Y\alpha = b$, we get a contradiction from $a = e\beta = e\lambda\eta\alpha = b$. Thus $S(X,Y) \notin BQ_m^n$ for $n = 1$ or 2. Finally, we assume that $n > 2$. Now, we show that $S(X,Y) \notin BQ_m^n$ for $m, n > 2$. Let

$$D = \left\{ \begin{pmatrix} Y & c & d & x \\ b & d & e & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}, \begin{pmatrix} a & Y \setminus \{a\} & c & d & x \\ a & b & c & b & e \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})} \right\}.$$

We see that

$$D^2 = \left\{ \begin{pmatrix} c & d & X \setminus \{c,d\} \\ e & a & b \end{pmatrix}, \begin{pmatrix} Y \cup \{c\} & d & x \\ b & e & a \end{pmatrix}_{x \in X \setminus (Y \cup \{c,d\})}, \right.$$

$$\begin{pmatrix} Y\cup\{d\} & c & x \\ b & d & a \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})}, \begin{pmatrix} a & Y\setminus\{a\} & c & d & x \\ a & b & c & b & e \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})} \Bigg\},$$

$$D^3 = D^2 \cup \left\{ \begin{pmatrix} c & X\setminus\{c\} \\ a & b \end{pmatrix}, \begin{pmatrix} d & X\setminus\{d\} \\ a & b \end{pmatrix}, \begin{pmatrix} c & X\setminus\{c\} \\ e & b \end{pmatrix}, \begin{pmatrix} A & X\setminus A \\ b & a \end{pmatrix}_{A=Y\cup\{c,d\}} \right\},$$

$$D^4 = D^3 \cup \left\{ \begin{pmatrix} X \\ b \end{pmatrix} \right\}$$

and $D^2 \subseteq D^3 \subseteq D^4 = D^5 = D^6 = \ldots$. Define $\beta = \begin{pmatrix} Y\cup\{c\} & d & x \\ b & d & a \end{pmatrix}_{X\setminus(Y\cup\{c,d\})}$. Then we have

$$\begin{aligned}
\beta =& \begin{pmatrix} Y\cup\{c\} & d & x \\ b & e & a \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})} \begin{pmatrix} a & Y\setminus\{a\} & X\setminus Y \\ a & b & d \end{pmatrix} \\
=& \begin{pmatrix} Y\cup\{c\} & d & x \\ b & c & e \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})} \begin{pmatrix} Y\cup\{d\} & c & x \\ b & d & a \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})} \\
\in& D^3 S(X,Y) \cap S(X,Y) D^3 \\
\subseteq& D^m S(X,Y) D \cap S(X,Y) D^n \subseteq (D)_{q(m,n)}.
\end{aligned}$$

If $S(X,Y) \in BQ_m^n$, then $\beta \in (D)_{q(m,n)} = (D)_{(m,n)}$. Since $\beta \notin \bigcup_{i\in\mathbb{N}} D^i$, $\beta = \lambda\eta\alpha$ for some $\lambda \in D^m, \eta \in S(X,Y), \alpha \in D^n$. Since $d \in X\beta$, we must have $\alpha = \begin{pmatrix} Y\cup\{d\} & c & x \\ b & d & a \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})}$. Since $D^m = D^4$ for all $m > 2$, we consider λ in 2 cases. Firstly, $\lambda = \begin{pmatrix} Y\cup\{c\} & d & x \\ b & e & a \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})}$, we have $a = e\beta = e\lambda\eta\alpha = a\eta\alpha = b$ because $a\eta \in Y$ which is a contradiction. Secondary, $\lambda \neq \begin{pmatrix} Y\cup\{c\} & d & x \\ b & e & a \end{pmatrix}_{x\in X\setminus(Y\cup\{c,d\})}$. Since $d\lambda \in Y$ for all λ and $d\lambda\eta \in Y$, we have $d = d\beta = d\lambda\eta\alpha = b$ which is a contradiction. Thus $S(X,Y) \notin BQ_m^n$.

Now, the proof is completed. $\square$