CHAPTER 1

Introduction

Let A be a nonempty set and $n \geq 1$ be a natural number. An n-ary operation on A is a function $f^A : A^n \to A$ and the natural number n is called the arity of f^A.

Let $(f_i^A)_{i\in I}$ be a sequence of n_i-ary operations on A indexed by a nonempty set I such that each n_i-ary operation f_i^A defined on A. Every n_i is called an arity of f_i^A.

Let $\tau := (n_i)_{i\in I}$ be the sequence of arities of f_i^A, $\tau = (n_i)_{i\in I}$ is called a type. The pair $\underline{A} = (A, (f_i^A)_{i\in I})$ is called an *algebra* of type $\tau = (n_i)_{i\in I}$. The set A is called the *base set* or *universe* of $\underline{A}$, and $(f_i^A)_{i\in I}$ is called the sequence of fundamental operations of $\underline{A}$. We use the notation $Alg(\tau)$ for the class of all algebras of a given type τ.

Varieties are collections of algebras which are classified by identities and hypervarieties are collections of algebras which are classified by hyperidentities. Hyperidentities and hypervarieties of a given type τ without nullary operations were originated by J. Aczel [1], V.D. Belousov [2], W. D. Neumann [9] and W. Taylor [15]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution. The notion of a hypersubstitution introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [5].

A hypersubstitution of type $\tau = (n_i)_{i\in I}$ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which maps each n_i-ary operation symbol of type $\tau = (n_i)_{i\in I}$ to an n_i-ary term of this type. We denote the set of all hypersubstitutions of type τ by $Hyp(\tau)$. We extend every hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ and defined a binary operation $\circ_h$ on $Hyp(\tau)$. It turns out that $(Hyp(\tau), \circ_h, \sigma_{id})$ is a monoid where $\sigma_1 \circ_h \sigma_2 := \hat{\sigma_1} \circ \sigma_2$, and σ_{id} is the identity element.

In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [8]. A generalized hypersubstitution of type $\tau = (n_i)_{i\in I}$ is a mapping σ which maps each n_i-ary operation symbol of type τ to a term of this type in $W_\tau(X)$ which does not necessarily preserve the arity. The set of all generalized hypersubstitutions of type τ denoted by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we need also the extension of each generalized hypersubstitution. The extension of such generalized hypersubstitution is defined as in the case of hypersubstitution. Then we can define a binary operation $\circ_G$ on $Hyp_G(\tau)$ where for any $\sigma_1, \sigma_2 \in Hyp_G(\tau), \sigma_1 \circ_G \sigma_2 := \hat{\sigma_1} \circ \sigma_2$. We conclude that $(Hyp(\tau), \circ_G, \sigma_{id})$ forms a monoid and $(Hyp(\tau), \circ_h, \sigma_{id})$ is a submonoid of $(Hyp_G(\tau), \circ_G, \sigma_{id})$.

This dissertation is organized into four chapters. Chapter 2 is to summarize basic concepts of some semigroup properties, generalized hypersubstitutions and several related backgrounds. Many algebraic properties of $Hyp_G(\tau)$ are investigated in Chapter 3. The main results are to determine all maximal completely regular submonoids of $Hyp_G(2)$ and apply the obtained results in $Hyp_G(2)$ to extend and determine all maximal completely regular submonoids of $Hyp_G(n)$. The summary of results obtained in this work are provided in Chapter 4.