CHAPTER 2

Preliminaries

The aim of this chapter is to briefly review some concepts of semigroup properties and some concepts of the monoid of all generalized hypersubstitutions that will be used throughout this thesis.

2.1 Semigroups

Let A be a nonempty set and $n \geq 1$ be a natural number. An n-ary operation on A is a function $f^A : A^n \to A$ and the natural number n is called the arity of f^A.

Let I be a nonempty indexed set, and let $(f_i^A)_{i \in I}$ be a function which assigns to every element of I an n_i-ary operation f_i^A defined on A. Then the pair $\underline{A} = (A, (f_i^A)_{i \in I})$ is called an *algebra*. The set A is called the *base set or universe of* $\underline{A}$, and $(f_i^A)_{i \in I}$ is called the *sequence of fundamental operations of* $\underline{A}$. The sequence $\tau := (n_i)_{i \in I}$ for all the arities is called the *type* of the algebra $\underline{A}$.

A groupoid $(S, \cdot)$ is defined as a nonempty set S together with a binary operation "$\cdot$" (by which we mean a map $\cdot := S \times S \to S$). We call $(S, \cdot)$ a semigroup if the operation $\cdot$ is associative, i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$. For convenience, we write ab instead of $a \cdot b$.

A semigroup S is called a *monoid* if S has an identity, i.e., there exists an element e in S such that $ae = a = ea$ for all $a \in S$. Clearly, if a semigroup has an identity, then that identity is unique.

Definition 2.1.1 ([6]). An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$.

Definition 2.1.2 ([10]). An element a of a semigroup S is called *completely regular* if there exists $x \in S$ such that $a = axa$ and $ax = xa$.

An element a of a semigroup S is called *left [right] regular* if $a \in Sa^2$ $[a \in a^2S]$. If all its elements are left regular [right regular] then a semigroup S is *left regular [right regular]*.

Theorem 2.1.3 ([10]). *An element a of a semigroup S is completely regular if and only if a is both left regular and right regular.*

Proof. Let a be a completely regular element in a semigroup S. Then there exists $x \in S$ such that $a = axa$ and $ax = xa$. So $a = axa = a^2x \in a^2S$ and $a = axa = xa^2 \in Sa^2$, i.e. a is both left regular and right regular.

Conversely, if a is both left regular and right regular element in a semigroup S, then $a \in a^2S \cap Sa^2$. There exist $x, y \in S$ such that $a = a^2x$ and $a = ya^2$. Consider

$$aya = ay(a^2x) = a(ya^2)x = aax = a^2x = a$$

$$axa = (ya^2)xa = y(a^2x)a = yaa = ya^2 = a$$

$$\textit{and} \quad ax = ya^2x = ya.$$

Hence $a(yax)a = (aya)xa = axa = a$ and $a(yax) = (aya)x = ax = ya = y(axa) = (yax)a$. Therefore a is completely regular. $\square$

Example 2.1.4. $(\mathbb{Z}_6, \cdot)$ is a commutative semigroup under multiplication with the identity $\overline{1}$. Consider

$$\overline{0} = \overline{0}^2 \cdot \overline{2} = \overline{2} \cdot \overline{0}^2, \qquad \overline{3} = \overline{3}^2 \cdot \overline{1} = \overline{1} \cdot \overline{3}^2$$

$$\overline{1} = \overline{1}^2 \cdot \overline{1} = \overline{1} \cdot \overline{1}^2, \qquad \overline{4} = \overline{4}^2 \cdot \overline{4} = \overline{4} \cdot \overline{4}^2$$

$$\overline{2} = \overline{2}^2 \cdot \overline{5} = \overline{5} \cdot \overline{2}^2, \qquad \overline{5} = \overline{5}^2 \cdot \overline{5} = \overline{5} \cdot \overline{5}^2.$$

So every element in $\mathbb{Z}_6$ is both left regular and right regular. By Theorem 2.1.3 we have that every element in $\mathbb{Z}_6$ is completely regular.

2.2 Permutations

Let $X_n = \{1, 2, ..., n\}$ which $n \geq 1$. A mapping $\pi : X_n \to X_n$ is called a *permutation* of X_n if π is both one-to-one and onto. The set of all permutations of X_n is denoted by S_n. It turns out that $(S_n, \circ)$ forms a group and is called the *symmetric group of degree n.*

To simplify the manipulation of these permutations, a matrix-typing notation is useful. For example, if the permutation $\pi : X_4 \to X_4$ is defined by $\pi(1) = 3$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 2$, we write it as

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

Here the image of each element of $X_n = \{1, 2, 3, 4\}$ is written below that element. In general, given $\pi \in S_n$ write it in matrix form as

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

Hence a typical member of S_n takes this form, where $\pi(1), \pi(2), ..., \pi(n)$ is the list of numbers $1, 2, ..., n$ in a (posibility) different order.

Example 2.2.1 ([7]). List the elements of S_3 in matrix notation.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

In particular, to construct a permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

we must choose the numbers $\pi(1), \pi(2), ..., \pi(n)$ from X_n so that they are all distinct. So we have n choices for $\pi(1)$, then $n - 1$ choices for $\pi(2)$, and so on. Thus π can be formed in $n(n-1)(n-2)...2 \cdot 1 = n!$ ways.

2.2.1 Cycles

Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 1 & 2 & 5 \end{pmatrix}$$

in S_6. Since the elements of π are moved in a cycle, π is called a *cycle* for this reason and we will write $\pi = (1\ 6\ 5\ 2\ 4)$. This notation lists only elements moved by π, and each is moved to its neighbor to the right, except the last element, which "cycle around" to the first. We generalize this type of permutation as follows.

Let $k_1, k_2, ..., k_r$ be distinct elements of X_n. Then the cycle $\pi = (k_1\ k_2\ ...\ k_r)$ is permutation in S_n defined as

$$\pi(k_i) = k_{i+1} \quad \text{if} \ \ 1 \leq i \leq r - 1$$

$$\pi(k_r) = k_1$$

$$\pi(k) = k \quad \text{if} \ \ k \notin \{k_1, k_2, ..., k_r\}.$$

We say that π has *length* r and refer to π as an *r-cycle*.

Example 2.2.2 ([7])**.** Let $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 5 & 7 & 9 & 14 & 10 & 11 & 12 & 8 & 3 & 13 & 2 & 6 & 4 & 1 \end{smallmatrix}\right)$ in cycle notation.
Solution. The cycle factorization of π is $\pi = (1\ 5\ 10\ 13\ 4\ 14)(2\ 7\ 12\ 6\ 11)(3\ 9)$.

Remark 2.2.3. *The only cycle of lenght 1 is the identity permutation* ε*.*

2.3 The Monoid of all Generalized Hypersubstitutions

In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [8]. The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid.

Let $X := \{x_1, x_2, ...\}$ be a countably infinite variables and $X_n := \{x_1, x_2, ..., x_n\}$ which $n \in \mathbb{N}$ is an n-element set. Let $\{f_i | i \in I\}$ be a set of n_i-ary operation symbols indexed by the set I. We call the sequence $\tau = (n_i)_{i \in I}$ of arities of f_i, the *type*. An n-ary term of type τ is defined inductively, as follows.

(i) Every $x_j \in X$ is an n-ary term of type τ.

(ii) If $t_1, t_2, ..., t_{n_i}$ are n-ary terms of type τ, then $f_i(t_1, t_2, ..., t_{n_i})$ is an n-ary term of type τ.

Example 2.3.1. Let $\tau = (3, 3)$. This means we have two ternary operation symbols, say f and g respectively. For some examples of ternary terms of type $(3, 3)$: $x_1, x_2, x_3, f(x_1, g(x_1, x_3, x_1), x_2), g(f(x_3, x_2, x_1), x_2, g(x_3, x_3, x_1)$.

The smallest set, which contains $x_1, x_2, ..., x_n$ and is closed under finite application of (ii), is denoted by $W_\tau(X_n)$ and it is called the set of all n-ary terms of type τ. It is clear that every n-ary term is also an m-ary term for all $m \geq n$. Let $W_\tau(X) = \cup_{n=1}^{\infty} W_\tau(X_n)$ be the set of all terms of type τ.

A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$, which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on the set of all generalized hypersubstitutions, we need the concept of a generalized superposition of terms and the extension of a generalized hypersubstitution, which are defined as follows.

Definition 2.3.2 ([8])**.** A generalized superposition of terms is a mapping
$S^n : W_\tau(X)^{n+1} \to W_\tau(X)$ such that

(i) $S^n(x_j, t_1, ..., t_n) = t_j$, if $1 \leq j \leq n$;

(ii) $S^n(x_j, t_1, ..., t_n) = x_j$, if $n < j$;

(iii) $S^n(t, t_1, ..., t_n) = f_i(S^n(s_1, t_1, ..., t_n), ..., S^n(s_{n_i}, t_1, ..., t_n))$, if $t = f_i(s_1, ..., s_{n_i})$.

We extend every generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ such that

(i) $\hat{\sigma}[x_j] = x_j \in X$;

(ii) $\hat{\sigma}[f_i(t_1, t_2, ..., t_{n_i})] = S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], ..., \hat{\sigma}[t_{n_i}])$ for any n_i-ary operation symbol f_i and suppose that $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, ..., x_{n_i})$.

Example 2.3.3. Let $\tau = (3, 2)$, i.e., we have one ternary operation symbol and one binary operation symbol, say that f and g, respectively. Let $\sigma : \{f, g\} \to W_{(3,2)}(X)$ where $\sigma(f) = f(x_1, g(x_3, x_1), x_2)$ and $\sigma(g) = g(x_2, x_3)$. Consider

$$\begin{aligned}
\hat{\sigma}[f(x_2, g(x_3, x_1), x_4)] &= S^3(\sigma(f), \hat{\sigma}[x_2], \hat{\sigma}[g(x_3, x_1)], \hat{\sigma}[x_4]) \\
&= S^3(\sigma(f), x_2, S^2(\sigma(g), \hat{\sigma}[x_3], \hat{\sigma}[x_2]), x_4) \\
&= S^3(\sigma(f), x_2, S^2(g(x_2, x_3), x_3, x_2), x_4) \\
&= S^3(f(x_1, g(x_3, x_1), x_2), x_2, g(x_2, x_3), x_4) \\
&= f(x_2, g(x_4, x_2), g(x_2, x_3)).
\end{aligned}$$

Example 2.3.4. Let $\tau = (2)$ with one binary operation f. Let $\sigma_1, \sigma_2 \in Hyp_G(2)$ where $\sigma_1(f) = f(x_3, f(x_2, x_5))$ and $\sigma_2(f) = f(f(x_4, x_1), f(x_2, x_3))$. Consider

$$\begin{aligned}
(\sigma_1 \circ_G \sigma_2)(f) &= \hat{\sigma_1}[f(f(x_4, x_1), f(x_2, x_3))] \\
&= S^2(\sigma_1(f), \hat{\sigma_1}[f(x_4, x_1)], \hat{\sigma_1}[f(x_2, x_3)]) \\
&= S^2(\sigma_1(f), S^2(\sigma_1(f), \hat{\sigma_1}[x_4], \hat{\sigma_1}[x_1]), S^2(\sigma_1(f), \hat{\sigma_1}[x_2], \hat{\sigma_1}[x_3])) \\
&= S^2(\sigma_1(f), S^2(f(x_3, f(x_2, x_5)), x_4, x_1), S^2(f(x_3, f(x_2, x_5)), x_2, x_3)) \\
&= S^2(f(x_3, f(x_2, x_5)), f(x_3, f(x_1, x_5)), f(x_3, f(x_3, x_5))) \\
&= f(x_3, f(f(x_3, f(x_3, x_5)), x_5)),
\end{aligned}$$

and

$$\begin{aligned}
(\sigma_2 \circ_G \sigma_1)(f) &= \hat{\sigma_2}[f(x_3, f(x_2, x_5))] \\
&= S^2(\sigma_2(f), \hat{\sigma_2}[x_3], \hat{\sigma_2}[f(x_2, x_5)]) \\
&= S^2(\sigma_2(f), x_3, S^2(\sigma_2(f), \hat{\sigma_2}[x_2], \hat{\sigma_2}[x_5])) \\
&= S^2(\sigma_2(f), x_3, S^2(f(f(x_4, x_1), f(x_2, x_3)), x_2, x_5)) \\
&= S^2(f(f(x_4, x_1), f(x_2, x_3)), x_3, f(f(x_4, x_2), f(x_5, x_3)))
\end{aligned}$$

$$= f(f(x_4, x_3), f(f(f(x_4, x_2), f(x_5, x_3)), x_3)).$$

Hence $\sigma_1 \circ_G \sigma_2 \neq \sigma_2 \circ_G \sigma_1$, i.e., $\circ_G$ is not commutative.

Let σ_{id} be the hypersubstitution which maps each n-ary operation symbol f_i to the term $f_i(x_1, ..., x_n)$. In 2000, S. Leeratanavalee and K. Denecke proved that for arbitrary terms $t, t_1, t_2, ..., t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2 \in Hyp_G(\tau)$, we have

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], ..., \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, ..., t_n)]$;

(ii) $(\sigma_1 \circ \sigma_2)\hat{} = \hat{\sigma_1} \circ \hat{\sigma_2}$.

Using the previous result, S. Leeratanavalee and K. Deneeke proved that $\underline{Hyp_G(\tau)} := (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid [8].