CHAPTER 3

Main Results

In semigroup theory, the main study approach is diverse some special elements in semigroups such as regular elements, quasi-regular elements and idempotent elements. In Chapter 2, we have that $(Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid. So we can characterize these special elements on this monoid. Th. Changphas and K. Denecke characterized idempotent elements and regular elements of the monoid of all hypersubstitutions of type τ [4]. W. Puninagool and S. Leeratanavalee [11] characterized the set of all regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$. In 2010, they characterized the set of all idempotent and regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (n)$ [12]. Furthermore, all idempotent and regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (3)$ were studied by S. Sudsanit and S. Leeratanavalee [13]. In 2013, S. Sudsanit, S. Leeratanavalee and W. Puninagool characterized left-right regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ [14].

The main results of this thesis are to characterize the set of all maximal completely regular submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ and determine all maximal completely regular submonoids of all generalized hypersubstitutions of type $\tau = (n)$.

Henceforth, we introduce some notations which will be used throughout of this thesis. For a type $\tau = (n)$ with an n-ary operation symbol f and $t \in W_{(n)}(X)$, we denote

$\sigma_t :=$ the generalized hypersubstitution of type $\tau = (n)$ which maps f to the term t,

$leftmost(t) :=$ the first variable (from the left) occurring in t,

$rightmost(t) :=$ the last variable occurring in t,

$var(t) :=$ the set of all variables occurring in the term t.

For a type $t \in W_{(n)}(X)$ and $1 \leq i \leq n$, an $i - most(t)$ is defined indeuctively by:

(i) if t is a variable, then $i - most(t) = t$,

(ii) if $t = f(t_1, ..., t_n)$, then $i - most(t) = i - most(t_i)$.

Notice that $1 - most(t) = leftmost(t)$ and $n - most(t) = rightmost(t)$.

Example 3.0.1. Let $\tau = (3)$ be a type, $t = f(x_2, f(x_8, x_5, x_3), f(x_1, x_6, x_4))$. Then $1 - most(t) = x_2$, $2 - most(t) = 2 - most(f(x_8, x_5, x_3)) = x_5$ and $3 - most(t) = 3 - most(f(x_1, x_6, x_4)) = x_4$.

3.1 All Maximal Completely Regular Submonoids of $Hyp_G(2)$

In the monoid of all generalized hypersubstitutions of type $\tau = (n)$, all regular elements were studied by W. Puninagool and S. Leeratanavalee in 2010 [12]. Moreover, in 2013, A. Boonmee and S. Leeratanavalee [3] characterized the set of all completely regular element of the monoid of all generalized hypersubstitutions of type $\tau = (n)$.

In this section, we used the concept of a completely regular element as a tool to determine the set of all maximal completely regular submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (n)$.

For a type $\tau = (n)$ with n-ary operation f, we denote:

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | t \in W_\tau(X) \backslash X \text{ and } var(t) \cap X_n = \varnothing\}$;

$R_3 := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{j_1}, ..., t_{i_m} = x_{j_m}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and for distinct $j_1, ..., j_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{j_1}, ..., x_{j_m}\}\}$;

$CR(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{\pi(i_1)}, ..., t_{i_m} = x_{\pi(i_m)}$ and π is a bijective map on $\{i_1, ..., i_m\}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, ..., x_{\pi(i_m)}\}\}$.

It is clearly that $CR(R_3) \subset R_3$. In 2010, W. Puninagool and S. Leeratanavalee [12] showed that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(n)$. In 2013, A. Boonmee and S. Leeratanavalee [3] determined the set of all completely regular elements in $Hyp_G(n)$, as the following theorems.

Theorem 3.1.1 ([3]). *For each $\sigma_t \in CR(R_3)$, σ_t is a completely regular element in $Hyp_G(n)$.*

Theorem 3.1.2 ([3]). *Let $CR(Hyp_G(n)) := CR(R_3) \cup R_1 \cup R_2$. Then $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$.*

Next, we will consider in case of $\tau = (2)$ this means that we have only one binary operation symbol, say that f, and then

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | t \in W_{(2)}(X) \backslash X \text{ and } var(t) \cap X_2 = \varnothing\}$;

$CR(R_3) := \{\sigma_t |\ t = f(t_1, t_2)$ where $t_i = x_i$ for some $i \in \{1, 2\}$ and $var(t) \cap X_2 = \{x_i\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$.

It is easily to see that $R_1, R_2, CR(R_3)$ are pairwise disjoint and $\underline{R_1}, \underline{R_2}$ are subsemigroups of $\underline{Hyp_G(2)}$ but $\underline{CR(R_3)}$ is not a submonoid of $\underline{Hyp_G(2)}$ as the following example.

Example 3.1.3. Let $\sigma_s, \sigma_t \in CR(R_3)$ such that $s = f(x_1, f(x_4, x_1))$ and $t = f(x_2, x_1)$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, f(x_4, x_1))]\\ &= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[f(x_4, x_1)])\\ &= S^2(\sigma_t(f), x_1, S^2(\sigma_t(f), \hat{\sigma}_t[x_4], \hat{\sigma}_t[x_1]))\\ &= S^2(\sigma_t(f), x_1, S^2(f(x_2, x_1), x_4, x_1))\\ &= S^2(\sigma_t(f), x_1, f(x_1, x_4))\\ &= S^2(f(x_2, x_1), x_1, f(x_1, x_4))\\ &= f(f(x_1, x_4), x_1).\end{aligned}$$

So $\sigma_t \circ_G \sigma_s \notin CR(R_3)$.

Next, let $\sigma_t \in Hyp_G(2)$, we denote

$CR_1(R_3) := \{\sigma_t | t = f(x_1, t') \textit{ where } t' \in W_{(2)}(X) \textit{ and } var(t) \cap X_2 = \{x_1\}\}$,

$CR_2(R_3) := \{\sigma_t | t = f(t', x_2) \textit{ where } t' \in W_{(2)}(X) \textit{ and } var(t) \cap X_2 = \{x_2\}\}$,

$CR'_1(R_3) := \{\sigma_t | t = f(x_1, t') \textit{ where } t' \in W_{(2)}(X),\ \ var(t) \cap X_2 = \{x_1\} \textit{ and } rightmost(t') \neq x_1\}$,

$CR'_2(R_3) := \{\sigma_t | t = f(t', x_2) \textit{ where } t' \in W_{(2)}(X)\ \ var(t) \cap X_2 = \{x_2\} \textit{ and } lefttmost(t') \neq x_2\}$,

$(MCR)_{Hyp_G(2)} := R_1 \cup R_2 \cup CR'_1(R_3) \cup CR'_2(R_3) \cup \{\sigma_{id}\}$,

$(MCR_1)_{Hyp_G(2)} := R_1 \cup R_2 \cup CR_1(R_3) \cup \{\sigma_{id}\}$,

$(MCR_2)_{Hyp_G(2)} := R_1 \cup R_2 \cup CR_2(R_3) \cup \{\sigma_{id}\}$ and

$(MCR_3)_{Hyp_G(2)} := R_1 \cup R_2 \cup \{\sigma_{id}, \sigma_{f(x_2,x_1)}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}\}$.

Proposition 3.1.4. *$\underline{CR_1(R_3) \cup \{\sigma_{id}\}}$ and $\underline{CR'_1(R_3) \cup \{\sigma_{id}\}}$ are submonoids of $\underline{Hyp_G(2)}$.*

Proof. It is clear that $CR_1(R_3) \subset Hyp_G(2)$. Next we show that $CR_1(R_3)$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in CR_1(R_3)$. Then $t = f(x_1, t'), s = f(x_1, s')$ where $t', s' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_1\}, var(s) \cap X_2 = \{x_1\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')]\\ &= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s'])\end{aligned}$$

$$
\begin{aligned}
&= S^2(f(x_1,t'),x_1,\hat{\sigma}_t[s']) \\
&= f(x_1,t') \qquad \text{since} \;\; var(t) \cap X_2 = \{x_1\} \\
&= \sigma_{f(x_1,t')}(f).
\end{aligned}
$$

Then $\sigma_t \circ_G \sigma_s \in CR_1(R_3)$ and therefore $\underline{CR_1(R_3) \cup \{\sigma_{id}\}}$ is a submonoid of $\underline{Hyp_G(2)}$. In case of $\underline{CR_1'(R_3) \cup \{\sigma_{id}\}}$ is a submonoid of $\underline{Hyp_G(2)}$, the proof is similar to the previous proof. $\square$

Proposition 3.1.5. *$\underline{CR_2(R_3) \cup \{\sigma_{id}\}}$ and $\underline{CR_2'(R_3) \cup \{\sigma_{id}\}}$ are submonoids of $\underline{Hyp_G(2)}$.*

Proof. It is clear that $CR_2(R_3) \subset Hyp_G(2)$. Next we show that $CR_2(R_3)$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in CR_2(R_3)$. Then $t = f(t',x_2), s = f(s',x_2)$ where $t', s' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_2\}, var(s) \cap X_2 = \{x_2\}$. Consider

$$
\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s',x_2)] \\
&= S^2(\sigma_t(f),\hat{\sigma}_t[s'],\hat{\sigma}_t[x_2]) \\
&= S^2(f(t',x_2),\hat{\sigma}_t[s'],x_2) \\
&= f(t',x_2) \qquad \text{since} \;\; var(t) \cap X_2 = \{x_2\} \\
&= \sigma_{f(t',x_2)}(f).
\end{aligned}
$$

Then $\sigma_t \circ_G \sigma_s \in CR_2(R_3)$ and therefore $\underline{CR_2(R_3) \cup \{\sigma_{id}\}}$ is a submonoid of $\underline{Hyp_G(2)}$. In case of $\underline{CR_2'(R_3) \cup \{\sigma_{id}\}}$ is a submonoid of $\underline{Hyp_G(2)}$, the proof is similar to the previous proof. $\square$

Theorem 3.1.6. *$\underline{(MCR)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$.*

Proof. By Theorem 3.1.2, we have every element in $(MCR)_{Hyp_G(2)}$ is completely regular. Next we show that $(MCR)_{Hyp_G(2)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR)_{Hyp_G(2)} = R_1 \cup R_2 \cup CR_1'(R_3) \cup CR_2'(R_3) \cup \{\sigma_{id}\}$.

Case 1: $\sigma_t \in R_1$. Then $t = x_i \in X$.

Case 1.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(2)}$.

Case 1.2: $\sigma_s \in R_2$. Then $s = f(s_1,s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. Consider

$$
\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1,s_2)] \\
&= S^2(\sigma_t(f),\hat{\sigma}_t[s_1],\hat{\sigma}_t[s_2]) \\
&= S^2(x_i,\hat{\sigma}_t[s_1],\hat{\sigma}_t[s_2])
\end{aligned}
$$

$$= \begin{cases} \hat{\sigma}_t[s_i], & \text{if } i \in \{1,2\}; \\ x_i, & \text{if } i > 2 \end{cases}$$

$$= \begin{cases} leftmost(s_1), & \text{if } i = 1; \\ rightmost(s_2), & \text{if } i = 2; \\ x_i, & \text{if } i > 2. \end{cases}$$

Then we have $\sigma_t \circ_G \sigma_s = \sigma_{x_j}$ for some $x_j \in X$. Hence $(\sigma_t \circ_G \sigma_s) \in R_1 \subset (MCR)_{Hyp_G(2)}$.

Case 1.3: $\sigma_s \in CR_1'(R_3)$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X), var(s) \cap X_2 = \{x_1\}$ and $rightmost(s') \neq x_1$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')] \\ &= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s']) \\ &= S^2(x_i, x_1, \hat{\sigma}_t[s']) \\ &= \begin{cases} x_1, & \text{if } i = 1; \\ \hat{\sigma}_t[s'] = rightmost(s'), & \text{if } i = 2; \\ x_i, & \text{if } i > 2. \end{cases} \end{aligned}$$

Then we have $\sigma_t \circ_G \sigma_s = \sigma_{x_j}$ for some $x_j \in X$. Hence $(\sigma_t \circ_G \sigma_s) \in R_1 \subset (MCR)_{Hyp_G(2)}$.

Case 1.4: $\sigma_s \in CR_2'(R_3)$. Then $s = f(s', x_2)$ where $s' \in W_{(2)}(X)$, $var(s) \cap X_2 = \{x_2\}$ and $leftmost(s') \neq x_2$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s', x_2)] \\ &= S^2(\sigma_t(f), \hat{\sigma}_t[s'], \hat{\sigma}_t[x_2]) \\ &= S^2(x_i, \hat{\sigma}_t[s'], x_2) \\ &= \begin{cases} x_2, & \text{if } i = 2; \\ \hat{\sigma}_t[s'] = leftmost(s'), & \text{if } i = 1; \\ x_i, & \text{if } i > 2. \end{cases} \end{aligned}$$

Then we have $\sigma_t \circ_G \sigma_s = \sigma_{x_j}$ for some $x_j \in X$. Hence $(\sigma_t \circ_G \sigma_s) \in R_1 \subset (MCR)_{Hyp_G(2)}$.

Case 2: $\sigma_t \in R_2$. Then $t \in W_{(2)}(X) \setminus X$ and $var(t) \cap X_2 = \varnothing$.

Case 2.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(2)}$.

Case 2.2: $\sigma_s \in R_2$. Then $s \in W_{(2)}(X) \setminus X$ and $var(s) \cap X_2 = \varnothing$. Since $\underline{R_2}$ is a subsemigroup of $\underline{Hyp_G(2)}$, so $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(2)}$.

Case 2.3: $\sigma_s \in CR_1'(R_3)$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X), var(s) \cap X_2 = \{x_1\}$ and $rightmost(s') \neq x_1$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')]\\ &= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s'])\\ &= S^2(f(t_1, t_2), x_1, \hat{\sigma}_t[s'])\\ &= f(t_1, t_2) \qquad \text{since } \ var(t) \cap X_2 = \varnothing.\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(2)}$.

Case 2.4: $\sigma_s \in CR_2'(R_3)$. Then $s = f(s', x_2)$ where $s' \in W_{(2)}(X), var(s) \cap X_2 = \{x_2\}$ and $leftmost(s') \neq x_2$. We can prove in the same manner as in Case 2.3.

Case 3: $\sigma_t \in CR_1'(R_3)$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X), var(t) \cap X_2 = \{x_1\}$ and $rightmost(t') \neq x_1$.

Case 3.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(2)}$.

Case 3.2: $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, s_2)]\\ &= S^2(\sigma_t(f), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2])\\ &= S^2(f(x_1, t'), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2])\\ &= f(\hat{\sigma}_t[s_1], t'') \qquad \text{since } \ var(t) \cap X_2 = \{x_1\},\end{aligned}$$

where t'' is a new term derived by substituting x_1 which occurs in t' by $\hat{\sigma}_t[s_1]$. Then $\sigma_t \circ_G \sigma_s \in R_2 \subseteq (MCR)_{Hyp_G(2)}$.

Case 3.3: $\sigma_s \in CR_1'(R_3)$. By Proposition 3.1.4., we have that $\underline{CR_1'(R_3)} \cup \underline{\{\sigma_{id}\}}$ is a submonoid of $\underline{Hyp_G(2)}$. So $\sigma_t \circ_G \sigma_s \in CR_1'(R_3) \subset (MCR)_{Hyp_G(2)}$.

Case 3.4: $\sigma_s \in CR_2'(R_3)$. Then $s = f(s', x_2)$ where $s' \in W_{(2)}(X), var(s) \cap X_2 = \{x_2\}$ and $leftmost(s') \neq x_2$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s', x_2)]\\ &= S^2(\sigma_t(f), \hat{\sigma}_t[s'], \hat{\sigma}_t[x_2])\end{aligned}$$

$$= S^2(f(x_1,t'),\hat{\sigma}_t[s'],x_2)$$
$$= f(\hat{\sigma}_t[s'],t''),$$

where t'' is a new term derived by substituting x_1 which occurs in t' by $\hat{\sigma}_t[s_1]$. Since $x_1 \notin var(s')$ and $leftmost(s') \neq x_2$, we have that $x_1, x_2 \notin var(\hat{\sigma}_t[s'])$. Since $x_2 \notin var(t')$ and $x_1, x_2 \notin var(\hat{\sigma}_t[s'])$, we have $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(2)}$.

Case 4: $\sigma_t \in CR_2'(R_3)$. Then $t = f(t', x_2)$ where $t' \in W_{(2)}(X), var(t) \cap X_2 = \{x_2\}$ and $leftmost(t') \neq x_2$. We can prove in the same manner as in Case 3. Therefore $\underline{(MCR)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$. □

Theorem 3.1.7. *$\underline{(MCR_1)_{Hyp_G(2)}}$ and $\underline{(MCR_2)_{Hyp_G(2)}}$ are completely regular submonoids of $\underline{Hyp_G(2)}$.*

Proof. By Theorem 3.1.2., we have every element in $(MCR_1)_{Hyp_G(2)}$ is completely regular. Next we show that $(MCR_1)_{Hyp_G(2)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR_1)_{Hyp_G(2)} = R_1 \cup R_2 \cup CR_1(R_3) \cup \{\sigma_{id}\}$.

Case 1: $\sigma_t \in R_1$. Then $t = x_i \in X$.

Case 1.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{Hyp_G(2)}$.

Case 1.2: $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1,s_2)] \\
&= S^2(\sigma_t(f),\hat{\sigma}_t[s_1],\hat{\sigma}_t[s_2]) \\
&= S^2(x_i,\hat{\sigma}_t[s_1],\hat{\sigma}_t[s_2]) \\
&= \begin{cases} \hat{\sigma}_t[s_i], & \text{if } i \in \{1,2\}; \\ x_i, & \text{if } i > 2 \end{cases} \\
&= \begin{cases} leftmost(s_1), & \text{if } i = 1; \\ rightmost(s_2), & \text{if } i = 2; \\ x_i, & \text{if } i > 2. \end{cases}
\end{aligned}$$

Then we have $\sigma_t \circ_G \sigma_s = \sigma_{x_j}$ for some $x_j \in X$. Hence $(\sigma_t \circ_G \sigma_s) \in R_1 \subset (MCR_1)_{Hyp_G(2)}$.

Case 1.3: $\sigma_s \in CR_1(R_3)$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ and $var(s) \cap X_2 = \{x_1\}$. Consider

$$(\sigma_t \circ_G \sigma_s)(f) = \hat{\sigma}_t[f(x_1,s')]$$

$$= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s'])$$
$$= S^2(x_i, x_1, \hat{\sigma}_t[s'])$$
$$= \begin{cases} x_1, & \text{if } \ i = 1; \\ \hat{\sigma}_t[s'] = rightmost(s'), & \text{if } \ i = 2; \\ x_i, & \text{if } \ i > 2. \end{cases}$$

Then we have $\sigma_t \circ_G \sigma_s = \sigma_{x_j}$ for some $x_j \in X$. Hence $(\sigma_t \circ_G \sigma_s) \in R_1 \subset (MCR_1)_{Hyp_G(2)}$.

Case 2: $\sigma_t \in R_2$. Then $t \in W_{(2)}(X) \setminus X$ and $var(t) \cap X_2 = \varnothing$.

Case 2.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{Hyp_G(2)}$.

Case 2.2: $\sigma_s \in R_2$. Then $s \in W_{(2)}(X) \setminus X$ and $var(s) \cap X_2 = \varnothing$. Since $\underline{R_2}$ is a subsemigroup of $\underline{Hyp_G(2)}$, so $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_1)_{Hyp_G(2)}$.

Case 2.3: $\sigma_s \in CR_1(R_3)$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ and $var(s) \cap X_2 = \{x_1\}$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')] \\ &= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s']) \\ &= S^2(f(t_1, t_2), x_1, \hat{\sigma}_t[s']) \\ &= f(t_1, t_2) \qquad \text{since } \ var(t) \cap X_2 = \varnothing. \end{aligned}$$

Then $\sigma_t \circ_G \sigma_S \in R_2 \subset (MCR_1)_{Hyp_G(2)}$.

Case 3: $\sigma_t \in CR_1(R_3)$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$, $var(t) \cap X_2 = \{x_1\}$.

Case 3.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{Hyp_G(2)}$.

Case 3.2: $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, s_2)] \\ &= S^2(\sigma_t(f), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2]) \\ &= S^2(f(x_1, t'), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2]) \\ &= f(\hat{\sigma}_t[s_1], t'') \qquad \text{since } \ var(t) \cap X_2 = \{x_1\}, \end{aligned}$$

where t'' is a new term derived by substituting x_1 which occurs in t' by $\hat{\sigma}_t[s_1]$. Then $\sigma_t \circ_G \sigma_s \in R_2 \subseteq (MCR_1)_{Hyp_G(2)}$.

Case 3.3: $\sigma_s \in CR_1(R_3)$. By Proposition 3.1.4., we have that $\underline{CR_1(R_3)} \cup$

$\underline{\{\sigma_{id}\}}$ is a submonoid of $\underline{Hyp_G(2)}$. So $\sigma_t \circ_G \sigma_s \in CR_1(R_3) \subset (MCR_1)_{Hyp_G(2)}$. Therefore $\underline{(MCR_1)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$. In case of $\underline{(MCR_2)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$, the proof is similar to the previous proof. $\square$

Theorem 3.1.8. *$\underline{(MCR_3)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$.*

Proof. By Theorem 3.1.2, we have every element in $(MCR_3)_{Hyp_G(2)}$ is completely regular. Next we show that $(MCR_3)_{Hyp_G(2)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR_3)_{Hyp_G(2)} = R_1 \cup R_2 \cup \{\sigma_{id}, \sigma_{f(x_2,x_1)}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}\}$.

Case 1: $\sigma_t \in R_1$. Then $t = x_i \in X$.

Case 1.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_2)_{Hyp_G(2)}$.

Case 1.2: $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. We can prove in the same manner as in Case 1.2 of Theorem 3.1.6. and conclude that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_3)_{Hyp_G(2)}$.

Case 1.3: $\sigma_s \in \{\sigma_{id}, \sigma_{f(x_2,x_1)}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}\}$. It is obvious that $\sigma_t \circ_G \sigma_s = \sigma_t$ if $\sigma_s = f(x_1, x_2)$. If $\sigma_s = f(x_2, x_1)$, then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_2, x_1)] \\
&= S^2(\sigma_t(f), \hat{\sigma}_t[x_2], \hat{\sigma}_t[x_1]) \\
&= S^2(x_i, x_2, x_1) \\
&= \begin{cases} x_2, & \text{if } \ i = 1; \\ x_1, & \text{if } \ i = 2; \\ x_i, & \text{if } \ i > 2. \end{cases}
\end{aligned}$$

If $\sigma_s = f(x_1, x_1)$ or $\sigma_s = f(x_2, x_2)$, we can prove similar to case of $\sigma_s = f(x_2, x_1)$. Then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_3)_{Hyp_G(2)}$.

Case 2: $\sigma_t \in R_2$. Then $t \in W_{(2)}(X) \setminus X$ and $var(t) \cap X_2 = \varnothing$. We can prove similar to Case 2 of Theorem 3.1.6., then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_3)_{Hyp_G(2)}$.

Case 3: $\sigma_t \in \{\sigma_{id}, \sigma_{f(x_2,x_1)}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}\}$.

Case 3.1: $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_3)_{Hyp_G(2)}$.

Case 3.2: $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. It is clear that $\sigma_t \circ_G \sigma_s = \sigma_s$, if $t = f(x_1, x_2)$. If $t = f(x_2, x_1)$, then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, s_2)] \\
&= S^2(\sigma_t(f), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2])
\end{aligned}$$

$$= S^2(f(x_2,x_1),\hat{\sigma_t}[s_1],\hat{\sigma_t}[s_2])$$
$$= f(\hat{\sigma_t}[s_2],\hat{\sigma_t}[s_1]).$$

Since $var(f(\hat{\sigma_t}[s_2],\hat{\sigma_t}[s_1])) \cap X_2 = \varnothing$, then $\sigma_{f(x_2,x_1)} \circ_G \sigma_s \in R_2 \subset (MCR_2)_{Hyp_G(2)}$. If $\sigma_s = f(x_1,x_1)$ or $\sigma_s = f(x_2,x_2)$, we can prove similar to case of $\sigma_s = f(x_2,x_1)$. Then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_3)_{Hyp_G(2)}$.

Case 3.3: $\sigma_s \in \{\sigma_{id}, \sigma_{f(x_2,x_1)}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}\}$.

If $\sigma_t = \sigma_{f(x_1,x_2)}$, then $\sigma_t \circ_G \sigma_s = \sigma_s \in \{\sigma_{id}, \sigma_{f(x_2,x_1)}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}\}$.

If $\sigma_t = \sigma_{f(x_2,x_1)}$, then

$$\sigma_t \circ_G \sigma_s = \begin{cases} \sigma_{f(x_2,x_1)}, & \text{if } s = f(x_1,x_2); \\ \sigma_{f(x_1,x_2)}, & \text{if } s = f(x_2,x_1); \\ \sigma_{f(x_1,x_1)}, & \text{if } s = f(x_1,x_1); \\ \sigma_{f(x_2,x_2)}, & \text{if } s = f(x_2,x_2). \end{cases}$$

If $\sigma_t = \sigma_{f(x_1,x_1)}$ or $\sigma_t = \sigma_{f(x_2,x_2)}$, we can prove similar to case of $\sigma_s = f(x_2,x_1)$. Then $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in \{\sigma_{id}, \sigma_{f(x_2,x_1)}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}\} \subset (MCR_3)_{Hyp_G}(2)$. Therefore, $\underline{(MCR_3)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$. □

Theorem 3.1.9. *$\underline{(MCR)_{Hyp_G(2)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(2)}$.*

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(2)}$ such that $(MCR)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$, then σ_t is completely regular.

Case 1: $\sigma_t \in CR_1(R_3) \backslash CR'_1(R_3)$. Choose $\sigma_s \in CR'_2(R_3)$, then $s = f(s',x_2)$ where $s' \in W_{(2)}(X), var(s) \cap X_2 = \{x_2\}$ and $leftmost(s') \neq x_2$. Consider

$$(\sigma_s \circ_G \sigma_t)(f) = \hat{\sigma_s}[f(x_1,t')]$$
$$= S^2(\sigma_s(f),\hat{\sigma_s}[x_1],\hat{\sigma_s}[t'])$$
$$= S^2(f(s',x_2),x_1,\hat{\sigma_s}[t'])$$
$$= f(s'',\hat{\sigma_s}[t']),$$

where s'' is a new term derived by substituting x_2 which occurs in s' by $\hat{\sigma_s}[t']$. Since $x_2 \in var(s)$ and $rightmost(t') = x_1$, we have $x_1 \in var(\hat{\sigma_s}[t'])$. By Theorem 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 2: $\sigma_t \in CR_2(R_3) \backslash CR'_2(R_3)$. Choose $\sigma_s \in CR'_1(R_3)$, then $s = f(x_1,s')$ where $s' \in W_{(2)}(X), var(s) \cap X_2 = \{x_1\}$ and $rightmost(s') \neq x_1$. Consider

$$(\sigma_s \circ_G \sigma_t)(f) = \hat{\sigma_s}[f(t',x_2)]$$

$$
\begin{aligned}
&= S^2(\sigma_s(f), \hat{\sigma_s}[t'], \hat{\sigma_s}[x_2]) \\
&= S^2(f(x_1, s'), \hat{\sigma_s}[t'], x_2) \\
&= f(\hat{\sigma_s}[t'], s'')
\end{aligned}
$$

where s'' is a new term derived by substituting x_1 which occurs in s' by $\hat{\sigma_s}[t']$. Since $x_1 \in var(s)$ and $leftmost(t') = x_2$, we have $x_2 \in var(\hat{\sigma_s}[t'])$. By Theorem 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 3: $\sigma_t = \sigma_{f(x_2,x_1)}$. Choose $\sigma_s \in CR_1'(R_3)$, then $s = f(x_1, s')$ where $s' \in W_{(2)}(X), var(s) \cap X_2 = \{x_1\}$ and $rightmost(s') \neq x_1$. Consider

$$
\begin{aligned}
(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(x_2, x_1)] \\
&= S^2(\sigma_s(f), \hat{\sigma_s}[x_2], \hat{\sigma_s}[x_1]) \\
&= S^2(f(x_1, s'), x_2, x_1) \\
&= f(x_2, s''),
\end{aligned}
$$

where s'' is a new term derived by substituting x_1 which occurs in s' by x_2. Since $x_2 \notin var(s')$ and the first input of the term $f(x_2, s')$ is x_2, by Theorem 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular. Therefore $K \subseteq (MCR)_{Hyp_G(2)}$ and thus $\underline{K} = \underline{(MCR)_{Hyp_G(2)}}$. ☐

Theorem 3.1.10. $\underline{(MCR_1)_{Hyp_G(2)}}$ *and* $\underline{(MCR_2)_{Hyp_G(2)}}$ *are maximal completely regular submonoids of* $\underline{Hyp_G(2)}$.

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(2)}$ such that $(MCR_1)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$, then σ_t is completely regular.

Case 1: $\sigma_t = \sigma_{f(x_2,x_1)}$. Choose $\sigma_s \in CR_1(R_3)$, then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ and $var(s) \cap X_2 = \{x_1\}$. Consider

$$
\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_1, s')] \\
&= S^2(\sigma_t(f), \hat{\sigma_t}[x_1], \hat{\sigma_t}[s']) \\
&= S^2(f(x_2, x_1), x_1, \hat{\sigma_t}[s']) \\
&= f(\hat{\sigma_t}[s'], x_1).
\end{aligned}
$$

Since $x_2 \notin var(t)$ and the second input of the term $f(\hat{\sigma_t}[s'], x_1)$ is x_1, by Theorem 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 2: $\sigma_t \in CR_2(R_3)$. Then $t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 =$

$\{x_2\}$. Choose $\sigma_s \in CR_1(R_3)$, then $s = f(x_1, s')$ where $s' \in W_{(2)}(X) \setminus X, var(s) \cap X_2 = \{x_1\}$ and $rightmost(s') = x_1$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')]\\ &= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s'])\\ &= S^2(f(t', x_2), x_1, \hat{\sigma}_t[s'])\\ &= f(t'', \hat{\sigma}_t[s']),\end{aligned}$$

where t'' is a new term derived by substituting x_2 which occurs in t' by $\hat{\sigma}_t[s']$. Since $x_2 \in var(t)$, we have $x_1 \in var(\hat{\sigma}_t[s'])$. Since $x_1 \notin var(t)$, by Theorem 3.1.2, we have $\sigma_t \circ_G \sigma_s$ is not completely regular. Then $\sigma_t \in (MCR_1)_{Hyp_G(2)}$. Therefore $K \subseteq (MCR_1)_{Hyp_G(2)}$ and thus $\underline{K} = \underline{(MCR_1)_{Hyp_G(2)}}$. In case of $\underline{(MCR_2)_{Hyp_G(2)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(2)}$, the proof is similar to the previous proof. □

Theorem 3.1.11. *$\underline{(MCR_3)_{Hyp_G(2)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(2)}$.*

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(2)}$ such that $(MCR_3)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$, then σ_t is a completely regular element.

Case 1: $\sigma_t \in CR_1(R_3)$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_1\}$. Choose $\sigma_s = \sigma_{f(x_2, x_1)}$, consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma}_s[f(x_1, t')]\\ &= S^2(\sigma_s(f), \hat{\sigma}_s[x_1], \hat{\sigma}_s[t'])\\ &= S^2(f(x_2, x_1), x_1, \hat{\sigma}_s[t'])\\ &= f(\hat{\sigma}_s[t'], x_1).\end{aligned}$$

Since $x_2 \notin var(t)$ and the second input of the term $f(\hat{\sigma}_s[t'], x_1)$ is x_1, by Theorem 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular. Then $\sigma_t \in (MCR_2)_{Hyp_G(2)}$.

Case 2: $\sigma_t \in CR_2(R_3)$. Then $t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_2\}$. We can prove in the same manner as in Case 1. Therefore $K \subseteq (MCR_3)_{Hyp_G(2)}$ and thus $\underline{K} = \underline{(MCR_3)_{Hyp_G(2)}}$. □

Corollary 3.1.12.

$$\{\underline{(MCR)_{Hyp_G(2)}}, \underline{(MCR_1)_{Hyp_G(2)}}, \underline{(MCR_2)_{Hyp_G(2)}}, \underline{(MCR_3)_{Hyp_G(2)}}\}$$

is the set of all maximal completely regular submonoids of $Hyp_G(2)$.

3.2 All Maximal Completely Regular Submonoids of $Hyp_G(n)$

In this section, we determine the set of all completely regular submonoids of the monoid of all generalized hypersubstitutions of type $\tau = (n)$. Denote R_1, R_2 and $CR(R_3)$ as in Section 3.1. By Theorem 3.1.2, we have that $R_1 \cup R_2 \cup CR(R_3)$ is the set of all completely regular elements in $Hyp_G(n)$.

Next, let $\sigma_t \in Hyp_G(n)$, we denote

$CR_1(R_3) := \{\sigma_t | t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}\}$.

$E := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_{i_1} = x_{i_1}, ..., t_{i_m} = x_{i_m}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and $var(t) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l\}$.

For any $\varnothing \neq I \subset \{1, ..., n\}$, let

$CR_I(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}$;$\forall\ i \in I$ and π is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i)} \mid \forall\ i \in I\}\}$.

$CR'_I(R_3) := \{\sigma_t | t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}$; $\pi(i) \in I$ for all $i \in I$ and $t_k = x_{\pi(k)}; \forall\ k \in \{1, ..., n\} \setminus I$ and π is a bijective map on $\{1, ..., n\}\}$.

We let

$(MCR)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_1(R_3)$,

$(MCR_1)_{Hyp_G(n)} := R_1 \cup R_2 \cup E$ and

$(MCR_I)_{Hyp_G(n)} := R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3) \cup \{\sigma_{id}\}$.

Theorem 3.2.1. *$(MCR)_{Hyp_G(n)}$ is a completely regular submonoid of $Hyp_G(n)$.*

Proof. By Theorem 3.1.2, we have every element in $(MCR)_{Hyp_G(n)}$ is completely regular. Next we show that $(MCR)_{Hyp_G(n)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR)_{Hyp_G(n)} = R_1 \cup R_2 \cup CR_1(R_3)$.

Case 1: $\sigma_t \in R_1$. Then $t = x_i \in X$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)] \\
&= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\
&= S^n(x_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) \\
&= \begin{cases} \hat{\sigma}_t[s_i], & \text{if } \ i \in \{1, ..., n\}; \\ x_i, & \text{if } \ i > n. \end{cases}
\end{aligned}$$

Since $t = x_i$, we have $\hat{\sigma_t}[s_i] = i - most(s_i) \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in CR_1(R_3)$, then $s = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_{\pi(1)}, ..., x_{\pi(n)})] \\
&= S^n(\sigma_t(f), \hat{\sigma_t}[x_{\pi(1)}], ..., \hat{\sigma_t}[x_{\pi(n)}]) \\
&= S^n(x_i, x_{\pi(1)}, ..., x_{\pi(n)}) \\
&= \begin{cases} x_{\pi(i)} & \text{,if } \ i \in \{1, ..., n\} \\ x_i & \text{,if } \ i > n. \end{cases}
\end{aligned}$$

So we have $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

Case 2: $\sigma_t \in R_2$. Then $t \in W_{(n)}(X) \setminus X$ and $var(t) \cap X_n = \varnothing$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in CR_1(R_3)$, then $s = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_{\pi(1)}, ..., x_{\pi(n)})] \\
&= S^n(\sigma_t(f), \hat{\sigma_t}[x_{\pi(1)}], ..., \hat{\sigma_t}[x_{\pi(n)}]) \\
&= S^n(f(t_1, ..., t_n), x_{\pi(1)}, ..., x_{\pi(n)}) \\
&= f(t_1, ..., t_n) \quad \text{since } \ var(t) \cap X_n = \varnothing.
\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(n)}$.

Case 3: $\sigma_t \in CR_1(R_3)$. Then $t = f(x_{\pi_1(1)}, ..., x_{\pi_1(n)})$ where π_1 is a bijective map on $\{1, ..., n\}$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, ..., s_n)] \\
&= S^n(\sigma_t(f), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\
&= S^n(f(x_{\pi(1)}, ..., x_{\pi(n)}), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\
&= f(w_1, ..., w_n) \quad \text{where } \ w_i = S^n(x_{\pi(i)}, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\
&\quad \text{for all } \ i \in \{1, .., n\}.
\end{aligned}$$

Since $var(\hat{\sigma_t}[s_i]) \cap X_n = \varnothing \ \ \forall i \in \{1, ..., n\}$, we have $var(f(w_1, ..., w_n)) \cap X_n = \varnothing$. Hence $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(n)}$.

If $\sigma_s \in CR_1(R_3)$, then $s = f(x_{\pi_2(1)}, ..., x_{\pi_2(n)})$ where π_2 is a bijective map on $\{1, ..., n\}$.
Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)]\\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= S^n(f(x_{\pi_1(1)}, ..., x_{\pi_1(n)}), x_{\pi_2(1)}, ..., x_{\pi_2(n)})\\ &= f(x_{\pi_2(\pi_1(1))}, ..., x_{\pi_2(\pi_1(n))})\\ &= f(x_{(\pi_2\circ\pi_1)(1)}, ..., x_{(\pi_2\circ\pi_1)(n)}).\end{aligned}$$

Since $\pi_1 \circ \pi_2$ is a bijective map on $\{1, ..., n\}$, we have $\sigma_t \circ_G \sigma_s \in CR_1(R_3)$. Therefore $\underline{(MCR)_{Hyp_G(n)}}$ is a completely regular submonoid of $\underline{Hyp_G(n)}$. $\square$

Theorem 3.2.2. *$\underline{(MCR_1)_{Hyp_G(n)}}$ is a completely regular submonoid of $\underline{Hyp_G(n)}$.*

Proof. By Theorem 3.1.2, we have every element in $(MCR_1)_{Hyp_G(n)}$ is completely regular. Next we show that $(MCR_1)_{Hyp_G(n)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR_1)_{Hyp_G(n)} = R_1 \cup R_2 \cup E$.

Case 1: $\sigma_t \in R_1$. Then $t = x_i \in X$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)]\\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= S^n(x_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= \begin{cases}\hat{\sigma}_t[s_i], & \text{if } \ i \in \{1, ..., n\};\\ x_i, & \text{if } \ i > n.\end{cases}\end{aligned}$$

Since $t = x_i$, we have $\hat{\sigma}_t[s_i] = i - most(s_i) \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{Hyp_G(n)}$.

If $\sigma_s \in E$, then $s = f(t_1, ..., s_n)$ where $s_{i_1} = x_{i_1}, ..., s_{i_m} = x_{i_m}$ for some $i_1, ..., i_m \in \{1, ..., n\}$ and $var(s) \cap X_n = \{x_{i_1}, ..., x_{i_m}\}$ and if $x_{i_l} \in var(s_k)$ for some $l \in \{1, ..., m\}$ and $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$, then $j - most(s_k) \neq x_{i_l}$ for all $j \neq i_l$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)]\\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= S^n(x_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\end{aligned}$$

$$= \begin{cases} \hat{\sigma}_t[s_i] & ,\text{if } \ i \in \{1,...,n\} \\ x_i & ,\text{if } \ i > n. \end{cases}$$

Since $t = x_i$, we have $\hat{\sigma}_t[s_i] = i - most(s_i) \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{HypG(n)}$.

Case 2: $\sigma_t \in R_2$. Then $t \in W_{(n)}(X) \setminus X$ and $var(t) \cap X_n = \varnothing$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{HypG(n)}$.

If $\sigma_s \in E$, then $s = f(t_1,...,s_n)$ where $s_{i_1} = x_{i_1},...,s_{i_m} = x_{i_m}$ for some $i_1,...,i_m \in \{1,...,n\}$ and $var(s) \cap X_n = \{x_{i_1},...,x_{i_m}\}$ and if $x_{i_l} \in var(s_k)$ for some $l \in \{1,...,m\}$ and $k \in \{1,...,n\} \backslash \{i_1,...,i_m\}$, then $j - most(s_k) \neq x_{i_l}$ for all $j \neq i_l$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1,...,s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_n],...,\hat{\sigma}_t[s_n]) \\ &= S^n(f(t_1,...,t_n), \hat{\sigma}_t[s_n],...,\hat{\sigma}_t[s_n]) \\ &= f(t_1,...,t_n) \quad \text{ since } \ var(t) \cap X_n = \varnothing. \end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_1)_{HypG(n)}$.

Case 3: $\sigma_t \in E$. Then $t = f(t_1,...,t_n)$ where $t_{i_1} = x_{i_1},...,t_{i_m} = x_{i_m}$ for some $i_1,...,i_m \in \{1,...,n\}$ and $var(t) \cap X_n = \{x_{i_1},...,x_{i_m}\}$ and if $x_{i_l} \in var(t_k)$ for some $l \in \{1,...,m\}$ and $k \in \{1,...,n\} \backslash \{i_1,...,i_m\}$, then $j - most(t_k) \neq x_{i_l}$ for all $j \neq i_l$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_1)_{HypG(n)}$.

If $\sigma_s \in R_2$, then $s = f(s_1,...,s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned} (\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1,...,s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1],...,\hat{\sigma}_t[s_n]) \\ &= S^n(f(t_1,...,t_n), \hat{\sigma}_t[s_1],...,\hat{\sigma}_t[s_n]) \\ &= f(w_1,...,w_n) \quad \text{ where } \ w_i = S^n(t_i, \hat{\sigma}_t[s_1],...,\hat{\sigma}_t[s_n]) \\ & \quad \text{for all } \ i \in \{1,..,n\}. \end{aligned}$$

Since $var(\hat{\sigma}_t[s_i]) \cap X_n = \varnothing \ \ \forall i \in \{1,...,n\}$, we have $var(f(w_1,...,w_n)) \cap X_n = \varnothing$. Hence $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{HypG(n)}$.

If $\sigma_s \in E$, then $s = f(s_1,...,s_n)$ where $s_{p_1} = x_{p_1},...,s_{p_{m'}} = x_{p_{m'}}$ for some $p_1,...,p_{m'} \in \{1,...,n\}$ and $var(s) \cap X_n = \{x_{p_1},...,x_{p_{m'}}\}$ and if $x_{p_{l'}} \in var(s_{k'})$ for some $l' \in \{1,...,m'\}$ and $k' \in \{1,...,n\} \backslash \{p_1,...,p_{m'}\}$, then $j' - most(s_{k'}) \neq x_{p_{l'}}$ for all $j' \neq p_{l'}$. Consider

$$(\sigma_t \circ_G \sigma_s)(f) = \hat{\sigma}_t[f(s_1,...,s_n)]$$

$$= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])$$

$$= S^n(f(t_1, ..., t_n), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])$$

$$= f(w_1, ..., w_n) \quad \text{where} \;\; w_i = S^n(t_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])$$

$$\text{for all} \;\; i \in \{1, .., n\}.$$

Case 1: $var(t_k) \cap X_n = \varnothing$ for all $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$ and $var(s_{k'}) \cap X_n = \varnothing$ for all $k' \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$.

Case 1.1: $i \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$. Then $w_i = S^n(t_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) = t_i$.

Case 1.2: $i \in \{i_1, ..., i_m\}$. Then $w_i = S^n(t_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) = \hat{\sigma}_t[s_i]$. If $i \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$, then $var(w_i) \cap X_n = \varnothing$. If $i \in \{p_1, ..., p_{m'}\}$, then $w_i = x_i$. By Case 1.1, 1.2, we have $\sigma_t \circ \sigma_s \in (R_2 \cup E) \subset (MCR_1)_{Hyp_G(n)}$.

Case 2: $var(t_k) \cap X_n = \varnothing$ for all $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$ and there exists $x_{p_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$, for all $k' \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$. It can be proved similarly as in Case 1. Hence $\sigma_t \circ \sigma_s \in (R_2 \cup E) \subset (MCR_1)_{Hyp_G(n)}$.

Case 3: There exists $x_{i_l} \in var(t_k)$ for some $l \in \{1, ..., m\}$, for all $k \in \{1, ..., n\} \backslash \{i_1, ..., i_m\}$ and there exists $x_{p_{l'}} \in var(s_{k'})$ for some $l' \in \{1, ..., m'\}$, for all $k' \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}\}$.

Case 3.1: $i \in \{i_1, ..., i_m\}$. Then $w_i = \hat{\sigma}_t[s_i]$.

For $i \in \{p_1, ..., p_{m'}\}$. Then $w_i = x_i$.

For $i = k'$. Then $w_i = \hat{\sigma}_t[s_i]$. If $i_l = p_{l'}$, then $w_{i_l} = x_{i_l}$. If $i_l \neq p_{l'}$, then $var(w_i) \cap \{x_{i_l}\} = \varnothing$.

For $i \in \{1, ..., n\} \backslash \{p_1, ..., p_{m'}, k'\}$. Then $var(w_i) \cap X_n = \varnothing$.

Case 3.2: $i \in \{1, ..., n\} \backslash \{i_1, ..., i_m, k\}$. Then $var(w_i) \cap X_n = \varnothing$. By Case 3.1,3.2, we have $\sigma_t \circ \sigma_s \in (R_2 \cup E) \subset (MCR_1)_{Hyp_G(n)}$. □

Theorem 3.2.3. *$\underline{(MCR_I)_{Hyp_G(n)}}$ is a completely regular submonoid of $\underline{Hyp_G(n)}$.*

Proof. By Theorem 3.1.2, we have every element in $(MCR_I)_{Hyp_G(n)}$ is completely regular. Next we show that $(MCR_I)_{Hyp_G(n)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR_I)_{Hyp_G(n)} = R_1 \cup R_2 \cup CR_I(R_3) \cup CR'_I(R_3) \cup \{\sigma_{id}\}$.

Case 1: $\sigma_t \in R_1$. Then $t = x_i \in X$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$(\sigma_t \circ_G \sigma_s)(f) = \hat{\sigma}_t[f(s_1, ..., s_n)]$$

$$= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])$$

$$= S^n(x_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])$$

$$= \begin{cases} \hat{\sigma_t}[s_i], & \text{if } \ i \in \{1, ..., n\}; \\ x_i, & \text{if } \ i > n. \end{cases}$$

Since $t = x_i$, we have $\hat{\sigma_t}[s_i] = i - most(s_i) \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in CR_I(R_3)$, then $s = f(s_1, ..., s_n)$ where $s_i = x_{\pi(i)}$; $\forall\, i \in I$ and π is a bijective map on I, $var(s) \cap X_n = \{x_{\pi(i)} \mid \forall\, i \in I\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= S^n(x_i, \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= \begin{cases} \hat{\sigma_t}[s_i], & \text{if } \ i \in \{1, ..., n\}; \\ x_i, & \text{if } \ i > n. \end{cases}\end{aligned}$$

Since $t = x_i$, we have $\hat{\sigma_t}[s_i] = i - most(s_i) \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in CR'_I(R_3)$, then $s = f(s_1, ..., s_n)$ where $s_i = x_{\pi(i)}$; $\pi(i) \in I$ for all $i \in I$, $s_k = x_{\pi(k)}$; $\forall\, k \in \{1, ..., n\} \setminus I$ and π is a bijective map on $\{1, ..., n\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_{\pi(1)}, ..., x_{\pi(n)})] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[x_{\pi(1)}], ..., \hat{\sigma_t}[x_{\pi(n)}]) \\ &= S^n(x_i, x_{\pi(1)}, ..., x_{\pi(n)}) \\ &= \begin{cases} x_{\pi(i)}, & \text{if } \ i \in \{1, ..., n\}; \\ x_i, & \text{if } \ i > n. \end{cases}\end{aligned}$$

Since $t = x_i$, then $x_{\pi(i)} = x_l$ for $l \in \{1, ..., n\}$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_I)_{HypG(n)}$.

Case 2: $\sigma_t \in R_2$. Then $t \in W_{(n)}(X) \setminus X$ and $var(t) \cap X_n = \varnothing$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in CR_I(R_3)$, then $s = f(s_1, ..., s_n)$ where $s_i = x_{\pi(i)}$; $\forall\, i \in I$ and π is a bijective map on I, $var(s) \cap X_n = \{x_{\pi(i)} \mid \forall\, i \in I\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, ..., s_n)] \\ &= S^n(\sigma_t(f), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= S^n(f(t_1, ..., t_n), \hat{\sigma_t}[s_1], ..., \hat{\sigma_t}[s_n]) \\ &= f(t_1, ..., t_n) \qquad \text{since } \ var(t) \cap X_n = \varnothing.\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in CR'_I(R_3)$, then $s = f(s_1, ..., s_n)$ where $s_i = x_{\pi(i)}$; $\pi(i) \in I$ for all $i \in I$, $s_k = x_{\pi(k)}$; $\forall\ k \in \{1, ..., n\} \setminus I$ and π is a bijective map on $\{1, ..., n\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_{\pi(1)}, ..., x_{\pi(n)})]\\ &= S^n(\sigma_t(f), \hat{\sigma}_t[x_{\pi(1)}], ..., \hat{\sigma}_t[x_{\pi(n)}])\\ &= S^n(f(t_1, ..., t_n), x_{\pi(1)}, ..., x_{\pi(n)})\\ &= f(t_1, ..., t_n) \quad \text{since } var(t) \cap X_n = \varnothing.\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_I)_{HypG(n)}$.

Case 3: $\sigma_t \in CR_I(R_3)$. Then $t = f(t_1, ..., t_n)$ where $t_i = x_{\pi_1(i)}$;$\forall\ i \in I$ and π_1 is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i)} \mid \forall\ i \in I\}$.

If $\sigma_s \in R_1$, then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in R_2$, then $s = f(s_1, ..., s_n)$ where $var(s) \cap X_n = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)]\\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= S^n(f(t_1, ..., t_n), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= f(w_1, ..., w_n) \quad \text{where } w_i = S^n(t_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &\quad \text{for all } i \in \{1, .., n\}.\end{aligned}$$

Since $var(\hat{\sigma}_t[s_i]) \cap X_n = \varnothing\ \ \forall i \in \{1, ..., n\}$, then $var(f(w_1, ..., w_n)) \cap X_n = \varnothing$. Hence $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_I)_{HypG(n)}$.

If $\sigma_s \in CR_I(R_3)$, then $s = f(s_1, ..., s_n)$ where $s_i = x_{\pi_2(i)}$;$\forall\ i \in I$ and π_2 is a bijective map on I, $var(s) \cap X_n = \{x_{\pi_2(i)} \mid \forall\ i \in I\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, ..., s_n)]\\ &= S^n(\sigma_t(f), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= S^n(f(t_1, ..., t_n), \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &= f(w_1, ..., w_n) \quad \text{where } w_i = S^n(t_i, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])\\ &\quad \text{for all } i \in \{1, .., n\}.\end{aligned}$$

For any $i_l \in I$, since π_1, π_2 are bijective maps on I there exist $i_p, i_q \in I$ such that $\pi_1(i_l) = i_p$ and $\pi_2(i_p) = i_q$. Then $w_{i_l} = S^n(t_{i_l}, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n]) = S^n(x_{\pi_1(i_l)}, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])$ $= \hat{\sigma}_t[s_{i_p}] = \hat{\sigma}_t[x_{\pi_2(i_p)}] = x_{i_q}$.

For any $j \in \{1, ..., n\} \setminus I$, let $t_j = f(u_1, ..., u_n)$. Consider

$$w_j = S^n(t_j, \hat{\sigma}_t[s_1], ..., \hat{\sigma}_t[s_n])$$

$$= S^n(f(u_1,..,u_n), \hat{\sigma}_t[s_1],...,\hat{\sigma}_t[s_n])$$
$$= f(w'_1,...,w'_n) \text{ where } w'_k = S^n(t_k, \hat{\sigma}_t[s_1],...,\hat{\sigma}_t[s_n])$$
$$\text{for all } k \in \{1,..,n\}.$$

If $var(u_k) \cap X_n = \varnothing$, then $w'_k = u_k$. If $u_k = x_{\pi_1(i_l)}$ and $\pi_1(i_l) = i_p, \pi_2(i_p) = i_q$, then $w'_k = S^n(u_k, \hat{\sigma}_t[s_1],...,\hat{\sigma}_t[s_n]) = S^n(x_{\pi_1(i_l)}, \hat{\sigma}_t[s_1],...,\hat{\sigma}_t[s_n]) = \hat{\sigma}_t[s_{i_p}] = x_{\pi_2(i_p)} = x_{i_q}; i_q \in I$. Hence $\sigma_t \circ_G \sigma_s \in CR_I(R_3) \subset (MCR_I)_{Hyp_G(n)}$.

If $\sigma_s \in CR'_I(R_3)$, we can prove as in the previous proof. Hence $\sigma_t \circ_G \sigma_s \in CR_I(R_3) \subset (MCR_I)_{Hyp_G(n)}$.

Case 4: $\sigma_t \in CR'_I(R_3)$. It can be proved similarly as in Case 3. Then we have $\sigma_t \circ_G \sigma_s \in (MCR_I)_{Hyp_G(n)}$. Therefore $\underline{(MCR_I)_{Hyp_G(n)}}$ is a completely regular submonoid of $\underline{Hyp_G(n)}$. $\square$

Theorem 3.2.4. $\underline{(MCR)_{Hyp_G(n)}}$ *is a maximal completely regular submonoid of* $\underline{Hyp_G(n)}$.

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(n)}$ such that $(MCR)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. Let $\sigma_t \in K$ where $\sigma_t \in CR(R_3) \setminus CR_1(R_3)$. Then $t = f(t_1,...,t_n)$ where $t_{i_l} = x_{\pi(i_l)}$; $\forall i_l \in I$ and π is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i_l)} \mid \forall\ i_l \in I\}$. Choose $\sigma_s \in CR_1(R_3)$ such that $s = f(x_{\pi'(1)},...,x_{\pi'(n)})$ where π' is a bijective map on $\{1,...,n\}$ and $\pi' = (\pi'(1)...\pi'(n))$ is a cycle. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_{\pi'(1)},...,x_{\pi'(n)})] \\
&= S^n(\sigma_t(f), \hat{\sigma}_t[x_{\pi'(1)}],...,\hat{\sigma}_t[x_{\pi'(n)}]) \\
&= S^n(f(t_1,...,t_n), x_{\pi'(1)},...,x_{\pi'(n)}) \\
&= f(w_1,...,w_n) \quad \text{where } w_j = S^n(t_j, x_{\pi'(1)},...,x_{\pi'(n)}) \\
&\quad \text{for all } j \in \{1,..,n\}.
\end{aligned}$$

Since $I \subset \{1,...,n\}$, there exists $i_p \in I$, $i_q \in \{1,...,n\} \setminus I$ such that $\pi'(i_p) = i_q$ and $\pi(i_l) = i_p, i_l, i_p \in I$, then

$$\begin{aligned}
w_{i_l} &= S^n(t_{i_l}, \hat{\sigma}_t[x_{\pi'(1)}],...,\hat{\sigma}_t[x_{\pi'(n)}]) \\
&= S^n(x_{\pi(i_l)}, x_{\pi'(1)},...,x_{\pi'(n)}) \\
&= x_{\pi'(i_p)} \\
&= x_{i_q}.
\end{aligned}$$

By Theorem 3.1.2, $\sigma_s \circ_G \sigma_t$ is not completely regular, so $\sigma_t \in (MCR)_{Hyp_G(n)}$. Therefore $K \subseteq (MCR)_{Hyp_G(n)}$ and thus $\underline{K} = \underline{(MCR)_{Hyp_G(n)}}$. $\square$

Theorem 3.2.5. $\underline{(MCR_1)_{Hyp_G(n)}}$ *is a maximal completely regular submonoid of* $\underline{Hyp_G(n)}$.

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(n)}$ such that $(MCR_1)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. Let $\sigma_t \in K$, then σ_t is a completely regular element.

Case 1: $\sigma_t \in CR_1(R_3)$. Then $t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is a bijective map on $\{1, ..., n\}$.

Case 1.1: $t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is bijective map on $\{1, ..., n\}$ and $(\pi(1)...\pi(n))$ is a cycle. Choose $\sigma_s \in E$ then $s = f(s_1, ..., s_n)$ where $s_{i_1} = x_{i_1}, ..., s_{i_m} = x_{i_m}$ and $s_j \in X \setminus X_n$, $\forall\ j \ \in \{1, ..., n\} \setminus \{i_1, ..., i_m\}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(x_{\pi(1)}, ..., x_{\pi(n)})]\\ &= S^n(\sigma_s(f), \hat{\sigma_s}[x_{\pi(1)}], ..., \hat{\sigma_s}[x_{\pi(n)}])\\ &= S^n(f(s_1, ..., s_n), x_{\pi(1)}, ..., x_{\pi(n)})\\ &= f(w_1, ..., w_n) \quad \text{where } \ w_j = S^n(s_j, x_{\pi(1)}, ..., x_{\pi(n)})\\ &\ \ \text{for all } \ j \in \{1, .., n\}.\end{aligned}$$

If $i_l \in \{i_1, ..., i_m\}$, then $w_{i_l} = x_{\pi(i_l)}$. Since $(\pi(1)...\pi(n))$ is a cycle, we have that $x_{\pi(i_l)} = x_{i_q}$; $i_q \in \{i_1, ...i_m\}\setminus\{i_l\}$. If $j \in \{1, ..., n\}\setminus\{i_1, ..., i_m\}$, then $w_j = s_j$. By Theorem 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 1.2: $t = f(x_{\pi(1)}, ..., x_{\pi(n)})$ where π is bijective map on $\{1, ..., n\}$ and there is $P = \{R_1, ..., R_l\}$ is a partition of $\{1, ..., n\}$ such that $R_1 = \{r_{11}, ..., r_{1f}\}, ..., R_l = \{r_{l1}, ..., r_{lh}\}$ and $(r_{11}...r_{1f})...(r_{l1}, ..., r_{lh})$. Let $d \in R_k$; $\exists\ \ k \in \{1, ..., l\}$ and $|R_k| > 1$. Choose $\sigma_s \in E$, then $s = f(s_1, ..., s_n)$ where $s_d = x_d$ and $s_q \in X \setminus X_n$, $\forall q \in \{1, ..., n\} \setminus \{d\}$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(x_{\pi(1)}, ..., x_{\pi(n)})]\\ &= S^n(\sigma_s(f), \hat{\sigma_s}[x_{\pi(1)}], ..., \hat{\sigma_s}[x_{\pi(n)}])\\ &= S^n(f(s_1, ..., s_n), x_{\pi(1)}, ..., x_{\pi(n)})\\ &= f(w_1, ..., w_n) \quad \text{where } \ w_j = S^n(s_j, x_{\pi(1)}, ..., x_{\pi(n)})\\ &\ \ \text{for all } \ j \in \{1, .., n\}.\end{aligned}$$

Then $w_j = x_{\pi(j)}$; $q \in \{1, ..., n\}\setminus\{i\}$. Since $d \in R_k$; $\exists\ \ k \in \{1, ..., l\}$ and $|R_k| > 1$, we have $x_{\pi(j)} = x_q$ and $w_i = s_i$, $i \in \{1, ..., n\}\setminus\{d\}$. By Theorem 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 2: $\sigma_t \in CR_I(R_3)\setminus E$. Then $t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}$;$\forall\ i \in I$ and π is a bijective map on I, $var(t) \cap X_n = \{x_{\pi(i)} \mid \forall \ \ i \in I\}$. Choose $\sigma_s \in E$ where $s = f(x_k, ..., x_k)$; $\exists\ \ k \in \{1, ..., n\}\setminus I$, $x_k \neq x_{\pi(i)}$; $\forall\ \ i \in I$. Consider

$$(\sigma_t \circ_G \sigma_s)(f) = \hat{\sigma_t}[f(x_k, ..., x_k)]$$

$$= S^n(\sigma_t(f), \hat{\sigma}_t[x_k], ..., \hat{\sigma}_t[x_k])$$

$$= S^n(f(t_1, ..., t_n), x_k, ..., x_k)$$

$$= f(w_1, ..., w_n) \quad \text{where} \;\; w_j = S^n(t_j, x_k, ..., x_k)$$

$$\text{for all} \;\; j \in \{1, .., n\}.$$

Then $w_k = t'_k$ where t'_k is a new term derived by substituting x_{i_l} $\forall$ $i_l \in I$ which occurs in t_k by x_k. By Theore 3.1.2, we have $\sigma_s \circ_G \sigma_t$ is not completely regular. Hence $\sigma_t \in (MCR_1)_{\underline{Hyp_G(n)}}$. Therefore $K \subseteq (MCR_1)_{\underline{Hyp_G(n)}}$ and thus $\underline{K} = \underline{(MCR_1)_{Hyp_G(n)}}$. $\square$

Theorem 3.2.6. $\underline{(MCR_I)_{Hyp_G(n)}}$ *is a maximal completely regular submonoid of* $\underline{Hyp_G(n)}$.

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $\underline{Hyp_G(n)}$ such that $(MCR_I)_{Hyp_G(n)} \subseteq K \subset Hyp_G(n)$. Let $\sigma_t \in K$ where $\sigma_t \in CR(R_3)\setminus(CR_I(R_3) \cup CR'_I(R_3))$ then $t = f(t_1, ..., t_n)$ where $t_i = x_{\pi(i)}$ and π is a bijective map on $\{1, ..., n\}$. Choose $\sigma_s \in CR_I(R_3)$ then $s = f(s_1, ..., s_n)$ where $s_i = x_{\pi'(i)}$;$\forall$ $i \in I$ and π' is a bijective map on I, $var(s) \cap X_n = \{x_{\pi'(i)} \mid \forall\, i \in I\}$ and $s_j \in X \setminus X_n$, $\forall$ j $\in \{1, ..., n\} \setminus I$. Consider

$$(\sigma_s \circ_G \sigma_t)(f) = \hat{\sigma_s}[f(x_{\pi(1)}, ..., x_{\pi(n)})]$$

$$= S^n(\sigma_s(f), \hat{\sigma_s}[x_{\pi(1)}], ..., \hat{\sigma_s}[x_{\pi(n)}])$$

$$= S^n(f(s_1, ..., s_n), x_{\pi(1)}, ..., x_{\pi(n)})$$

$$= f(w_1, ..., w_n) \quad \textit{where} \;\; w_j = S^n(s_j, x_{\pi(1)}, ..., x_{\pi(n)})$$

$$\text{for all} \;\; j \in \{1, .., n\}.$$

Since $I \subset \{1, ..., n\}$ there exist $i_p \in I$, $i_q \in \{1, ..., n\} \setminus I$ such that $\pi(i_p) = i_q$ and $\pi'(i_r) = i_p$; $i_r, i_p \in I$, Then

$$w_{i_r} = S^n(s_{i_r}, x_{\pi(1)}, ..., x_{\pi(n)})$$

$$= S^n(x_{i_p}, x_{\pi(1)}, ..., x_{\pi(n)})$$

$$= x_{\pi(i_p)}$$

$$= x_{i_q}.$$

By Theorem 3.1.2, $\sigma_s \circ_G \sigma_t$ is not completely regular, so $\sigma_t \in (MCR_I)_{Hyp_G(n)}$. Therefore $K \subseteq (MCR_I)_{Hyp_G(n)}$ and thus $\underline{K} = \underline{(MCR_I)_{Hyp_G(n)}}$. $\square$

Corollary 3.2.7. $\{\underline{(MCR)_{Hyp_G(n)}}, \underline{(MCR_1)_{Hyp_G(n)}}\} \cup \{\underline{(MCR_I)_{Hyp_G(n)}} \mid \varnothing \neq I \subset \{1, ..., n\}\}$ *is the set of all maximal completely regular submonoids of* $\underline{Hyp_G(n)}$.