CHAPTER 2

Basic Concepts and Preliminaries

In this chapter, we recall and give some useful definitions, and results which will be used in the later chapter.

2.1 Metric Spaces

Definition 2.1.1. [13] Let X be a nonempty set. A metric on X is a function $D : X \times X \to [0, \infty)$ which satisfies the following conditions

(MS1) $D(x, y) \geq 0$ for each $x, y \in X$, and $D(x, y) = 0$ if and only if $x = y$,

(MS2) $D(x, y) = D(y, x)$ for each $x, y \in X$,

(MS3) $D(x, y) \leq D(x, z) + D(z, y)$ for each $x, y, z \in X$.

Then D is called a *distance* or *metric* on X, and X together with D is called a *metric space* which will be denoted by (X, D).

Example 2.1.2. [13]

1. Let $D(x, y) = |x - y|$, for all $x, y \in \mathbb{R}$, where $| \cdot |$ denotes the absolute value. Then D is a metric on $\mathbb{R}$. The metric D is called the *usual metric* for $\mathbb{R}$.

2. The Euclidian space $\mathbb{R}^n$ with
$$D(x, y) = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{\frac{1}{2}},$$
for each $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, is a metric space. The metric D is called the *Euclidian metric for* $\mathbb{R}^n$. The next two mappings
$$\rho(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|, \text{ and } \sigma(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$
for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ are also metrics on $\mathbb{R}^n$.

3. Let X be a nonempty set and $x, y \in X$. Let $D : X \times X \to \mathbb{R}$ be defined by
$$D(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then (X, D) is a metric space which is called a *discrete space.*

4. Let X be the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. We define a metric D by
$$D(f, g) = \max_{x \in [a,b]} |f(x) - g(x)| \text{ for all } f, g \in X.$$
Then (X, D) is a metric space and usually denoted by $C[a, b]$.

Definition 2.1.3. [13] Let (X, D) be a metric space. A sequence $\{x_n\} \subset X$ is said to be

- *convergent* if there exists $x \in X$ such that $\lim_{n \to \infty} D(x_n, x) = 0$,
- *Cauchy* if for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $D(x_m, x_n) < \epsilon$ for all $m, n > N$.
- The metric spaces (X, D) is *complete* if all Cauchy sequences in X is convergent in X.

Note: From now on, we denote that the sequence $\{x_n\}$ converges to a point $x \in X$ by $x_n \to x$ as $n \to \infty$.

Definition 2.1.4. [13] Let $(X, D), (Y, D')$ be metric spaces. We say that $T : X \to Y$ is *continuous at* $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $D'(Tx, Tx_0) < \epsilon$ for all $x \in X$ satisfying $D(x, x_0) < \delta$. Moreover, T is said to be *continuous* if it is continuous at every point of X.

Theorem 2.1.5. *[13] Let $(X, D), (Y, D')$ be metric spaces. $T : X \to Y$ is* continuous at *$x_0 \in X$ if and only if $x_n \to x_0$ as $n \to \infty \Rightarrow Tx_n \to Tx_0$ as $n \to \infty$.*

2.2 Generalized α-ψ-Geraghty Contraction Type Mapping

In 2014, Karapinar [12] presented the existence of fixed point generalized α-ψ-Geraghty contraction type mapping in complete metric space. First, we recall the definition of auxiliary functions.

Definition 2.2.1. [12] Let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy these conditions

(ψ1) ψ is nondecreasing,

(ψ2) ψ is continuous,

(ψ3) ψ is subadditive, that is, $\psi(s + t) \le \psi(s) + \psi(t)$ for all $s, t \in [0, \infty)$,

($\psi 4$) $\psi(t) = 0$ if and only if $t = 0$.

Example 2.2.2. For $t \in [0, \infty)$, $\psi(t) = \frac{t}{2}$ is a good example for $\psi \in \Psi$ since it satisfies all conditions $(\psi 1) - (\psi 4)$. Indeed, for $t_1, t_2 \in [0, \infty)$

($\psi 1$) if $t_1 \leq t_2$, then $\psi(t_1) = \frac{t_1}{2} \leq \frac{t_2}{2} = \psi(t_2)$,

($\psi 2$) given $\epsilon > 0$, there exists $\delta = 2\epsilon$ such that $|t_1 - t_2| < \delta$ implies

$$|\psi(t_1) - \psi(t_2)| = |\tfrac{t_1}{2} - \tfrac{t_1}{2}| = \tfrac{1}{2}|t_1 - t_2| < \tfrac{\delta}{2} = \epsilon,$$

($\psi 3$) $\psi(t_1 + t_2) = \frac{t_1+t_2}{2} = \frac{t_1}{2} + \frac{t_2}{2} = \psi(s) + \psi(t)$,

($\psi 4$) $\psi(t_1) = \frac{t_1}{2} = 0$ if and only if $t = 0$.

The following class of functions was defined by Geraghty [8].

Definition 2.2.3. Let $\mathcal{F}$ be the family of all functions $\beta : [0, \infty) \to [0, 1)$ such that for any $\{t_n\} \subset [0, \infty)$

$$\text{if } \lim_{n\to\infty} \beta(t_n) = 1 \text{ then } \lim_{n\to\infty} t_n = 0.$$

Example 2.2.4. For $t \in [0, \infty)$, $\beta(t) = 2^t$ is a good example of $\beta \in \mathcal{F}$ since for any $\{t_n\} \subset [0, \infty)$,

$$\lim_{n\to\infty} 2^{t_n} = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Now, we recall the notion of generalized α-ψ-Geraghty contraction type.

Definition 2.2.5. [12] Let (X, D) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A mapping $T : X \to X$ is said to be *a generalized α-ψ-Geraghty contraction type* if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x, y)\psi(D(Tx, Ty)) \leq \beta(\psi(M(x, y))) \cdot \psi(M(x, y)), \tag{2.1}$$

where $M(x, y) = \max\{D(x, y), D(x, Tx), D(y, Ty)\}$ and $\psi \in \Psi$.

Example 2.2.6. Let $X = [0, \infty)$ and $D(x, y) = |x - y|$ for all $x, y \in X$. Then (X, D) is a metric space. Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{1}{3}x, & \text{if } x \in [0, 1], \\ 3x, & \text{otherwise.} \end{cases}$$

Also, we define a function $\alpha : X \times X \to \mathbb{R}$ in a following way

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1], \\ \\ 0, & \text{otherwise.} \end{cases}$$

If $x \notin [0,1]$ or $y \notin [0,1]$, then it is obvious that the inequality (2.1) holds.

If $x, y \in [0,1]$, then let $\beta(t) = \frac{1}{3}$ and $\psi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. It is clear that $\beta \in \mathcal{F}$ and $\psi \in \Psi$. Consider

$$\begin{aligned} &\beta(\psi(M(x,y))) \cdot \psi(M(x,y)) - \alpha(x,y)\psi(D(Tx,Ty)) \\ &= \frac{1}{3} \cdot \frac{M(x,y)}{2} - \frac{|x-y|}{6} \\ &\geq \frac{M(x,y)}{6} - \frac{M(x,y)}{6} \\ &= 0. \end{aligned}$$

Hence, for $x, y \in [0,1]$, the inequality (2.1) holds. Therefore T is a generalized α-ψ-Geraghty contraction type.

Definition 2.2.7. [12] Let $T : X \to X$ be a self mapping on a nonempty set X and $\alpha : X \times X \to \mathbb{R}$. We say that T is *triangular α-admissible* if these conditions hold

1. for any $x, y \in X$, if $\alpha(x,y) \geq 1$ then $\alpha(Tx,Ty) \geq 1$,
2. for any $x, u, y \in X$, if $\alpha(x,u) \geq 1$ and $\alpha(u,y) \geq 1$ then $\alpha(x,y) \geq 1$.

Example 2.2.8. Let $T(x) = x^2$ for $x \in [0,\infty)$. Define $\alpha : X \times X \to \mathbb{R}$ by:

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1] \\ \\ 0 & \text{otherwise.} \end{cases}$$

Let $x, y \in X$. We can see that if $\alpha(x,y) \geq 1$, then $x, y \in [0,1]$. Thus $x^2, y^2 \in [0,1]$ which implies that $\alpha(Tx,Ty) = \alpha(x^2,y^2) \geq 1$.

If $\alpha(x,u) \geq 1$ and $\alpha(u,y) \geq 1$, then $x, u, y \in [0,1]$ which implies that $\alpha(x,y) \geq 1$.

Thus T is triangular α-admissible.

Proposition 2.2.9. *[16] Let X be a nonempty set and $T : X \to X$ be triangular α-admissible. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ such that $n < m$.*

Proposition 2.2.10. *Let X be a nonempty set and $T : X \to X$ be triangular α-admissible. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, x_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ such that $n \leq m$.*

Proof. By Proposition 2.2.9, $\alpha(x_n, x_m) \geq 1$ for all m, n such that $n < m$. From hypothesis, we have $\alpha(x_0, x_0) \geq 1$. Assume that $\alpha(x_n, x_n) \geq 1$. By triangular α-admissible of T, $\alpha(x_{n+1}, x_{n+1}) = \alpha(Tx_n, Tx_n) \geq 1$. Thus, by mathematical induction, $\alpha(x_n, x_n) \geq 1$ for all $n \in \mathbb{N}$. $\square$

Definition 2.2.11. [12] Let $\alpha : X \times X \to \mathbb{R}$ be a function. A sequence $\{x_n\}$ in X is said to be *α-regular* if it satisfies the following condition

If $\{x_n\}$ is a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.

In 2014, Karapinar proved the following result in his paper.

Example 2.2.12. [12] Let (X, D) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ and $T : X \to X$ be a mapping. If these conditions hold

1. T is a generalized α-ψ-Geraghty contraction type map,

2. T is triangular α-admissible,

3. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

4. either T is continuous or $\{x_n\}$ is α-regular.

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

The first objective of this thesis is to establish fixed point theorem of generalized α-ψ-Geraghty in an RS-generalized metric spaces which is an extension of Theorem 2.2.12.

2.3 Rectangular Metric Spaces and B_N-Spaces

In 2000, Branciari [3] generalized metric space by replacing the triangle inequality with a weaker condition.

Definition 2.3.1. Let X be a nonempty set and $N \in \mathbb{N}$. A mapping $D : X \times X \to [0, \infty)$ is said to be a *Branciari N-metric* if it fits to these conditions

(BN1) $D(x, y) = 0$ if and only if $x = y$, (BN2) $D(x,y) = D(y,x)$,

(BN3) $D(x,y) \leq D(x,z_1) + D(z_1,z_2) + \cdots + D(z_{N-1},z_N) + D(z_N,y)$

for all $x, y, z_1, z_2, \ldots, z_{N-1}, z_N \in X$ are all different.

The pair (X,D) is called a B_N-space. If $N = 2$, then the pair (X,D) is called a rectangular metric space.

Example 2.3.2. Let $A = \{0,2\}, B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define $D : X \times X \to [0,1]$ by

$$D(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x,y \in A \text{ or } x,y \in B, \\ y, & \text{if } x \in A \text{ and } y \in B. \end{cases}$$

A pair (X,D) is a rectangular metric space.

It is easy to see that the condition (BN1) and (BN2) hold. For the last condition of the rectangular metric, let $x, y, z_1, z_2 \in X$ and distinct from each other. If $x = y$ then it is obvious that (BN3) holds. If $x \neq y$ and x, y are in the same set then

$$D(x,y) = 1 \leq D(x,z_1) + D(z_1,z_2) + D(z_2,y).$$

Because $D(z_1,z_2) = 1$ if z_1, z_2 are in the same set and $D(x,z_1) = 1$ or $D(z_2,y) = 1$ if z_1, z_2 are in the different set. If $x \neq y$ and x, y are in the different set, assume that $x \in A$ and $y \in B$ then

$$D(x,y) = y \leq 1 \leq D(x,z_1) + D(z_1,z_2) + D(z_2,y).$$

Because $D(z_1,z_2) = 1$ if z_1, z_2 are in the same set and $D(x,z_1) = 1$ or $D(z_2,y) = 1$ if z_1, z_2 are in the different set. Therefore, (X,D) is a rectangular metric space.

2.4 JKS Contraction

In 2014, Jleli, Karapinar and Samet [10] defined an interesting class of auxiliary functions to extend the Bananch contraction principle to the wider class of mapping.

Definition 2.4.1. [10] Let Θ be the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions

(θ1) θ is nondecreasing,

($\theta 2$) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0,$$

($\theta 3$) there exists $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l$,

($\theta 4$) θ is continuous.

Example 2.4.2. For $t \in (0, \infty)$, $\theta(t) = e^{t^{\frac{1}{2}}}$ is an example for $\theta \in \Theta$ since it satisfies all conditions as follows

($\theta 1$) For any $a, b \in (0, \infty)$ such that $a \le b$, we have $\theta(a) = e^{a^{\frac{1}{2}}} \le e^{b^{\frac{1}{2}}} = \theta(b)$.

($\theta 2$) For each sequence $\{t_n\} \subset (0, \infty)$,

$$\begin{aligned} \lim_{n\to\infty} \theta(t_n) = \lim_{n\to\infty} e^{t_n^{\frac{1}{2}}} = 1 &\iff \ln(\lim_{n\to\infty} e^{t_n^{\frac{1}{2}}}) = \ln(1) \\ &\iff \lim_{n\to\infty} \ln(e^{t_n^{\frac{1}{2}}}) = 0 \\ &\iff \lim_{n\to\infty} t_n^{\frac{1}{2}} = 0 \\ &\iff \lim_{n\to\infty} t_n = 0. \end{aligned}$$

($\theta 3$) Given $r = \frac{1}{2}$, we have

$$\lim_{t\to 0^+} t^{\frac{1}{2}} = 0 \text{ and } \lim_{t\to 0^+} (e^{t^{\frac{1}{2}}} - 1) = 0.$$

Using L'Hospital's rule, we will obtain

$$l = \lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = \lim_{t\to 0^+} \frac{e^{t^{\frac{1}{2}}} - 1}{t^{\frac{1}{2}}} = \lim_{t\to 0^+} \frac{e^{t^{\frac{1}{2}}} \cdot \frac{d}{dt}(t^{\frac{1}{2}})}{\frac{d}{dt}(t^{\frac{1}{2}})} = \lim_{t\to 0^+} e^{t^{\frac{1}{2}}} = \infty.$$

($\theta 4$) Since exponential and square root function are continuous, its composition must be continuous.

After Jleli, Karapinar and Samet introduced the new class of mapping, they proved the following theorem in the setting of rectangular metric space.

Example 2.4.3. [10] Let (X, D) be a complete rectangular metric space and $T : X \to X$ be a given map. Suppose that there exists $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$x, y \in X, D(Tx, Ty) \neq 0 \Rightarrow \theta(D(Tx, Ty)) \le (\theta(M(x, y)))^k$$

where $M(x, y) = \max\{D(x, y), D(x, Tx), D(y, Ty)\}$. Then T has a unique fixed point.

The second purpose of this thesis is to extend Theorem 2.4.3 to the framework of RS-generalized metric space.

2.5 Generalized Metric Spaces

In 1985, Matthews [14] was the first one who studied metric domain.

Definition 2.5.1. Let X be a nonempty set. A mapping $D : X \times X \to [0, \infty)$ is said to be a metric domain if it fits to these conditions for all $x, y, z \in X$

(DS1) if $D(x, y) = 0$ then $x = y$,

(DS2) $D(x, y) = D(y, x)$,

(DS3) $D(x, y) \leq D(x, z) + D(z, y)$.

The pair (X, D) is called a *dislocated metric space.*

Later, Hitzler and Seda [9] defined its topology and renamed it to dislocated metric. The notion of dislocated topologies has useful applications in context of logic programming semantics.

Example 2.5.2. Let $X = \mathbb{R}$ and $D(x, y) = x^2 + y^2$ for all $x, y \in X$. Then (X, D) is a dislocated metric space. Indeed, for all $x, y, z \in X$

(DS1) If $D(x, y) = x^2 + y^2 = 0$ then $x = y = 0$. However, the converse does not hold since $D(1, 1) = 1^2 + 1^2 = 2 \neq 0$.

(DS2) $D(x, y) = x^2 + y^2 = y^2 + x^2 = D(y, x)$.

(DS3) $D(x, y) = x^2 + y^2 \leq (x^2 + z^2) + (z^2 + y^2) = D(x, z) + D(z, y)$.

In 1993 and 1998, Czerwik [6, 7] presented this new class of spaces as follows.

Definition 2.5.3. Let X be a nonempty set. A mapping $D : X \times X \to [0, \infty)$ is said to be a b-metric if it fits to these conditions for all $x, y, z \in X$

(BS1) $D(x, y) = 0$ if and only if $x = y$,

(BS2) $D(x, y) = D(y, x)$,

(BS3) there exists $s \geq 1$ such that $D(x, y) \leq s(D(x, z) + D(z, y))$.

The pair (X, D) is called a b-metric space.

Example 2.5.4. Let $X = \mathbb{R}$ and $D(x, y) = (x - y)^2$ for all $x, y \in X$. We will show that (X, D) is a b-metric space with $s = 2$. Let $x, y, z \in \mathbb{R}$ then

(BS1) $D(x,y) = (x-y)^2 = 0 \iff (x-y) = 0 \iff x = y$

(BS2) $D(x,y) = (x-y)^2 = (y-x)^2 = D(y,x)$.

(BS3) Since $0 \leq ((x-z)-(z-y))^2$,

$$0 \leq (x-z)^2 - 2(x-z)(z-y) + (z-y)^2.$$

We have that

$$2(x-z)(z-y) \leq (x-z)^2 + (z-y)^2 = D(x,z) + D(z,y).$$

Consider

$$\begin{aligned} D(x,y) &= (x-y)^2 \\ &= ((x-z)+(z-y))^2 \\ &= (x-z)^2 + 2(x-z)(z-y) + (z-y)^2 \\ &\leq 2[(x-z)^2 + (z-y)^2] \\ &= 2[D(x,z) + D(z,y)]. \end{aligned}$$

Therefore, (X, D) is a b-metric space but it is not a metric space since for $x = 1, y = -2$ and $z = \frac{1}{2}$, we have

$$D(x,y) = 9 > 6\tfrac{1}{2} = \tfrac{1}{4} + \tfrac{25}{4} = D(x,z) + D(z,y).$$

In 2015, Jleli and Samet [11] introduced a new class of generalized metric called JS-generalized metric space.

Definition 2.5.5. Let X be a space equipped with generalized metric D and $x \in X$. A sequence $\{x_n\} \subseteq X$ is said to be

1. D-Cauchy if $\lim\limits_{m,n\to\infty} D(x_m, x_n) = 0$,
2. D-converge to x if $\lim\limits_{n\to\infty} D(x_n, x) = 0$.

Definition 2.5.6. Let X be a nonempty set. A mapping $D : X \times X \to [0, \infty]$ is said to be *a JS-generalized metric* if it fits to these conditions for all $x, y \in X$

(JS1) if $D(x,y) = 0$ then $x = y$,

(JS2) $D(x,y) = D(y,x)$,

(JS3) there exists $K > 0$ such that if $(x, y) \in X \times X$ and x_n D-converges to x then,

$$D(x, y) \leq K \limsup_{n\to\infty} D(x_n, y).$$

The pair (X, D) is called *a JS-generalized metric space.*

Example 2.5.7. Let $X = [0, \infty]$. Define $D(x, y) = (x+y)^2$ where $x, y < \infty$ and $D(x, y) = \infty$ where x or $y = \infty$. Then, (X, D) is a JS-generalized metric space with $K = 1$. Indeed, for all $x, y \in X$

(JS1) if $D(x, y) = (x + y)^2 = 0$ then $x = y = 0$. The converse doesn't hold. The example is $D(1, 1) = (1 + 1)^2 = 4 \neq 0$.

(JS2) $D(x, y) = (x + y)^2 = (y + x)^2 = D(y, x)$.

(JS3) Let $\{x_n\} \subseteq X$ such that $\lim_{n\to\infty} D(x_n, x) = 0$, that is

$$0 = \lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} (x_n + x)^2$$

This implies that $x = 0$ and $\lim_{n\to\infty} x_n = 0$.

Since $D(x, y) = D(0, y) = y^2 \leq (x_n + y)^2 = D(x_n, y)$ for all $n \in \mathbb{N}$, we have

$$D(x, y) \leq (1) \limsup_{n\to\infty} D(x_n, y).$$

This JS-generalized metric space has a limit uniqueness property.

Proposition 2.5.8. *[11] Let (X, D) be a JS-generalized metric space. Let $\{x_n\}$ be a sequence in X and $x, y \in X$. If $\{x_n\}$ D-converges to x and y, then $x = y$.*

In 2016, Roldán and Shahzad [17] introduced the new class of generalized metric space called RS-generalized metric space.

Definition 2.5.9. Let X be a nonempty set, $x_0 \in X$ and let $T : X \to X$. The sequence $\{x_n\}$ in X is said to be

1. infinite if $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ such that $n \neq m$.

2. almost periodic if there exists $n_0 \in \mathbb{N}$ and $N \in \mathbb{N}$ such that

$$x_{n_0+r+Nk} = x_{n_0+r} \text{ for all } k \in \mathbb{N} \text{ and } r \in \{0, 1, 2, ..., N-1\}$$

3. Picard if $x_n = T^n x_0$ for all $n \in \mathbb{N}$.

Denote the set of all elements in the Picard sequence as $O_T(x_0)$ where x_0 is the starter of the sequence.

Proposition 2.5.10. *[17] Every Picard sequence is either infinite or almost periodic.*

Definition 2.5.11. Let $(X, D), (Y, D')$ be generalized metric spaces. T is said to be *continuous* if for any sequences $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.5.12. [17] Let X be a nonempty set and $D : X \times X \to [0, \infty]$ be a function. We call a pair (X, D) *RS-generalized metric space* if the function D satisfies these conditions

(RS1) for all $x, y \in X$. If $D(x, y) = 0$ then $x = y$,

(RS2) for all $x, y \in X$, $D(x, y) = D(y, x)$,

(RS3) there exists $C > 0$ such that if $x, y \in X$ are two points and $\{x_n\}$ is a D - Cauchy infinite sequence in X such that $\{x_n\} \to x$ in D then

$$D(x, y) \leq C \limsup_{n\to\infty} D(x_n, y).$$

The space is said to be complete if for any D-Cauchy sequence D-converges to a point in X.

Remark 2.5.13. *This class of space covers standard metric space, b-metric space, dislocated metric space, JS-generalized metric space, B_N-space.*

Example 2.5.14. [17] Let (Y, d) be a B_N-space contain a sequence that D-converges to two different points and $Z = \{z_0, z_1, \ldots, z_N, z_{N+1}\}$ be a set with $N + 2$ members disjoints from Y. Let $X = Y \cup Z$ and define $D : X \times X \to [0, \infty]$ by

$$D(x,y) = \begin{cases} d(x,y), & \text{if } x, y \in Y \\ \infty, & \text{if } x \in Y \text{ and } y \in Z \text{ or viceversa} \\ 0, & \text{if } x = y \in Z \\ 3, & \text{if } x \neq y \text{ and } z_1 \in \{x, y\} \\ M, & \text{if } x, y = \{z_0, z_{N+1}\} \\ 2, & \text{otherwise } (x, y \in Z \text{ and } x \neq y) \end{cases}$$

where $2 \leq M \leq 3N + 2$. Then (X, D) is a real RS-generalized space. Indeed for $x, y \in X$

(RS1) If $D(x,y)=0$, then $d(x,y)=0$ or $x=y\in Z$. Both cases lead to $x=y$.

(RS2) We can easily see that $D(x,y)=D(y,x)$.

(RS3) Now, let $\{x_n\}$ be a D-Cauchy infinite sequence such that $x_n \to x$ as $n\to\infty$. Since $\{x_n\}$ is infinite and $Z\cup\{x,y\}$ is a finite set, for $\epsilon>0$ we can find $k\in\mathbb{N}$ such that

$$x_n \in X\backslash(Z\cup\{x,y\}) = Y\backslash\{x,y\} \text{ and } D(x_n,x)<\epsilon \text{ for all } n\geq k$$

This implies that $x\in Y$.

Case 1: If $y\notin Y$, then for all $C>0$

$$D(x,y)=\infty = C\limsup_{n\to\infty} D(x_n,y).$$

Case 2: If $y\in Y$, we have $D(x_n,y)=d(x_n,y)$ for all $n\geq k$. Since (Y,d) is a B_N-space, we have

$$d(x,y)\leq d(x,x_{k+1})+d(x_{k+1},x_{k+2})+\cdots+d(x_{k+N-1},x_{k+N})+d(x_{k+N},y).$$

Taking $\limsup$ over k both sides, since $\{x_n\}$ D-Cauchy and D-converge to x as $n\to\infty$, then we have

$$D(x,y)=d(x,y)\leq (1)\limsup_{k\to\infty} d(x_k,y)=\limsup_{k\to\infty} D(x_k,y).$$

This shows that (X,D) is an *RS*-generalized space. Since there exists a sequence with two different limits in Y, (X,D) is not a JS-generalized space. Since $D(x,y)$ can be infinity, (X,D) is not a B_N-space.

Although the limit of convergent sequences is not unique, we still have the following property.

Proposition 2.5.15. *[17] Let $\{x_n\}$ be a D-Cauchy sequence in an RS-generalized metric space (X,D) such that $\{x_n\}$ is infinite or Picard. If $\{x_n\}$ D-converges to x and y in X, then $x=y$.*

Definition 2.5.16. [17] Let $\{x_n\}$ be a Picard sequence in X with $T:X\to X$ and $x_0\in X$. We define

$$\delta_k(D,T,x_0)=\sup_{a,b\geq k}\{D(x_a,x_b)\}.$$

For $k=0$, we denote $\delta(D,T,x_0)$ instead of $\delta_0(D,T,x_0)$.

Example 2.5.17. Given $X = [0,1]$ and $D(x,y) = |x-y|$ for all $x, y \in X$. Then (X, D) is a metric space. Let $x_0 = 1$ and $T(x) = \frac{x}{2}$ for all $x \in X$. For $k \in \mathbb{N}$, we have $\delta_k\{D, T, x_0\} = \sup\limits_{a,b \geq k} \{D(x_a, x_b)\} = \dfrac{1}{2^k}$ and $\delta(D, T, x_0) = 1$.