CHAPTER 3

Main Results

In this chapter, we study fixed point theorems for generalized α-ψ-Geraghty contraction type and JKS contraction in complete RS-generalized metric spaces.

3.1 Generalized α-ψ-Geraghty Contraction Type

In this section, we present the existence and the uniqueness of fixed points of generalized $\alpha - \psi$-Geraghty contraction type in RS-generalized metric space. Now, we are ready to state the first result.

Theorem 3.1.1. *Let (X, D) be a complete RS-generalized metric space, $\alpha : X \times X \to \mathbb{R}$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold*

1. *T is a generalized α-ψ-Geraghty contractive type mapping,*
2. *T is triangular α-admissible,*
3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(x_0, x_0) \geq 1$ and $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*
4. *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. If $x_{n'} = x_{n'+1}$ for some $n' \in \mathbb{N}$, then $x_{n'}$ is a fixed point of T. Thus, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. This implies that $D(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$.

First, we will show that $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$. By Proposition 2.2.10, we know that

$$\alpha(x_i, x_j) \geq 1 \text{ for all } i \leq j.$$

For $n > n_0$

$$\begin{aligned} \psi(D(x_{n+1}, Dx_{n+2})) &= \psi(D(Tx_n, Tx_{n+1})) \\ &\leq \alpha(x_n, x_{n+1})\psi(D(Tx_n, Tx_{n+1})) \\ &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1})), \end{aligned} \tag{3.1}$$

where

$$M(x_n, x_{n+1}) = \max\{D(x_n, x_{n+1}), D(x_n, Tx_n), D(x_{n+1}, Tx_{n+1})\}$$
$$= \max\{D(x_n, x_{n+1}), D(x_{n+1}, x_{n+2})\}.$$

If $M(x_n, x_{n+1}) = D(x_{n+1}, x_{n+2})$, we have

$$\psi(D(x_{n+1}, x_{n+2})) \leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1}))$$
$$= \beta(\psi(D(x_{n+1}, x_{n+2})))\psi(D(x_{n+1}, x_{n+2}))$$
$$< \psi(D(x_{n+1}, x_{n+2}))$$

which is a contradiction. It follows that $M(x_n, x_{n+1}) = D(x_n, x_{n+1})$ and

$$\psi(D(x_{n+1}, x_{n+2})) \leq \beta(\psi(D(x_n, x_{n+1})))\psi(D(x_n, x_{n+1})) < \psi(D(x_n, x_{n+1})). \tag{3.2}$$

Since ψ is nondecreasing, if we suppose that $D(x_{n+1}, x_{n+2}) > D(x_n, x_{n+1})$. Then we have $\psi(D(x_{n+1}, x_{n+2})) \geq \psi(D(x_n, x_{n+1}))$ which is a contradiction to (3.2). Therefore, we have $D(x_{n+1}, x_{n+2}) \leq D(x_n, x_{n+1})$, for all $n \geq n_0$.

Since a sequence $\{D(x_n, x_{n+1})\}_{n \geq n_0}$ is nonincreasing and bounded below. Therefore $\lim_{n\to\infty} D(x_n, x_{n+1}) = \epsilon$ for some $\epsilon \geq 0$. Suppose that $\epsilon > 0$, from (3.2), we have

$$\frac{\psi(D(x_{n+1}, x_{n+2}))}{\psi(D(x_n, x_{n+1}))} \leq \beta(\psi(D(x_n, x_{n+1}))) < 1. \tag{3.3}$$

By definition of ψ, we have

$$\lim_{n\to\infty} \psi(D(x_{n+1}, x_{n+2})) = \psi(\lim_{n\to\infty} D(x_{n+1}, x_{n+2})) = \psi(\epsilon) > 0$$

and

$$\lim_{n\to\infty} \psi(D(x_n, x_{n+1})) = \psi(\lim_{n\to\infty} D(x_n, x_{n+1})) = \psi(\epsilon) > 0.$$

Taking limit over n in (3.3), we have

$$1 \leq \lim_{n\to\infty} \beta(\psi(D(x_n, x_{n+1}))) \leq 1.$$

Thus, we can conclude that

$$\lim_{n\to\infty} \beta(\psi(D(x_n, x_{n+1}))) = 1.$$

By property of β, we have

$$\lim_{n\to\infty} \psi(D(x_n, x_{n+1})) = 0.$$

By property of ψ, we have

$$\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$$

as a contradiction. We thus conclude that $\lim_{n\to\infty} D(x_n, x_{n+1}) = 0$.

Now, we claim that $\{x_n\}$ is Cauchy. Suppose on the contrary that there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}, \{x_{n_k}\}$ such that $m_k \geq n_k \geq k$ and m_k is the smallest index for which

$$D(x_{n_k}, x_{m_k}) \geq \epsilon \text{ for all } k \in \mathbb{N}.$$

Consider when $k \geq n_0$, we have

$$\begin{aligned}
\psi(\epsilon) &\leq \psi(D(x_{n_k}, x_{m_k})) \\
&= \psi(D(Tx_{n_k-1}, Tx_{m_k-1})) \\
&\leq \alpha(x_{n_k-1}, x_{m_k-1})\psi(D(x_{n_k-1}, x_{m_k-1})) \\
&\leq \beta(\psi(M(x_{n_k-1}, x_{m_k-1})))\psi(M(x_{n_k-1}, x_{m_k-1})) \\
&< \psi(M(x_{n_k-1}, x_{m_k-1}))
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n_k-1}, x_{m_k-1}) &= \max\{D(x_{n_k-1}, x_{m_k-1}), D(x_{n_k-1}, Tx_{n_k-1}), D(x_{m_k-1}, Tx_{m_k-1})\} \\
&= \max\{D(x_{n_k-1}, x_{m_k-1}), D(x_{n_k-1}, x_{n_k}), D(x_{m_k-1}, x_{m_k})\}.
\end{aligned}$$

If $M(x_{n_k-1}, x_{m_k-1}) \neq D(x_{n_k-1}, x_{m_k-1})$, we have

$$\psi(\epsilon) \leq \lim_{k\to\infty} \psi(M(x_{n_k-1}, x_{m_k-1})) = \psi(0).$$

Since ψ is nondecreasing, we have $\epsilon \leq 0$ which is a contradiction. Therefore

$$M(x_{n_k-1}, x_{m_k-1}) = D(x_{n_k-1}, x_{m_k-1}) \text{ for all } k \geq n_0.$$

This implies that, for all $k \geq n_0$

$$\begin{aligned}
\psi(D(x_{n_k}, x_{m_k})) &= \psi(D(Tx_{n_k-1}, x_{m_k-1})) \\
&\leq \alpha(x_{n_k-1}, x_{m_k-1})\psi(D(x_{n_k-1}, x_{m_k-1})) \\
&\leq \beta(\psi(D(x_{n_k-1}, x_{m_k-1})))\psi(D(x_{n_k-1}, x_{m_k-1})).
\end{aligned}$$

Repeating this argument, we have

$$\begin{aligned}
\psi(\epsilon) &\leq \psi(D(x_{n_k}, x_{m_k})) \\
&\leq \beta(\psi(D(x_{n_k-1}, x_{m_k-1}))) \cdot \psi(D(x_{n_k-1}, x_{m_k-1})) \\
&\leq \beta(\psi(D(x_{n_k-1}, x_{m_k-1}))) \cdot \beta(\psi(D(x_{n_k-2}, x_{m_k-2}))) \cdot \psi(D(x_{n_k-2}, x_{m_k-2}))
\end{aligned}$$

$$
\begin{aligned}
&\leq \cdots \\
&\leq \prod_{i=0}^{n_k-n_0} \beta(\psi(D(x_{n_0+i}, x_{m_k-(n_k-n_0)+i}))) \cdot \psi(D(x_{n_0}, x_{m_k-(n_k-n_0)})) \\
&\leq \prod_{i=0}^{n_k-n_0} \beta(\psi(D(x_{n_0+i}, x_{m_k-(n_k-n_0)+i})))\psi(\delta_{n_0}(D, T, x_0)).
\end{aligned}
$$

For each $k \geq n_0$, define

$$
\beta_k = \beta(\psi(D(x_{n_0+i_k}, x_{m_k-(n_k-n_0)+i_k}))) = \max_{0 \leq i \leq n_k-n_0} \{\beta(\psi(D(x_{n_0+i}, x_{m_k-(n_k-n_0)+i})))\}.
$$

Thus, for $k \geq n_0$

$$
\psi(\epsilon) \leq \psi(D(x_{n_k}, x_{m_k})) \leq \beta_k^{n_k-n_0} \psi(\delta_{n_0}(D, T, x_0)).
$$

If $\limsup_{k\to\infty} \beta_k < 1$, then $\lim_{k\to\infty} \beta_k^{n_k-n_0} = 0$ which is a contradiction. If $\limsup_{k\to\infty} \beta_k = 1$, by passing through a subsequence, we have

$$
1 = \lim_{k\to\infty} \beta_k = \lim_{k\to\infty} \beta(\psi(D(x_{n_0+i_k}, x_{m_k-(n_k-n_0)+i_k}))).
$$

This implies that

$$
\lim_{k\to\infty} \psi(D(x_{n_0+i_k}, x_{m_k-(n_k-n_0)+i_k})) = 0.
$$

Thus, there exists $k_0 \in \mathbb{N}$ such that

$$
\psi(D(x_{n_{k_0}+i_{k_0}}, x_{m_{k_0}-(n_{k_0}-n_0)+i_{k_0}})) < \psi(\tfrac{\epsilon}{2}).
$$

Therefore,

$$
\begin{aligned}
\psi(\epsilon) &\leq \psi(D(x_{n_{k_0}}, x_{m_{k_0}})) \\
&\leq \prod_{i=1}^{i_{k_0}} \beta(\psi(D(x_{n_{k_0}+k_0}, x_{m_{k_0}-(n_{k_0}-n_0)+i}))) \cdot \psi(D(x_{n_{k_0}+i_{k_0}}, x_{m_{k_0}-(n_{k_0}-n_0)+i_{k_0}})) \\
&< \psi(D(x_{n_{k_0}+i_{k_0}}, x_{m_{k_0}-(n_{k_0}-n_0)+i_{k_0}})) < \psi(\frac{\epsilon}{2})
\end{aligned}
$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X, D) is complete, then $x_n \to z \in X$ as $n \to \infty$.

By continuity of T, $Tx_n \to Tz$ as $n \to \infty$. By Theorem 2.5.15, we have $z = Tz$ as required.

□

Lemma 3.1.2. *From Theorem 3.1.1, if z is a fixed point of T, $D(z, Tz) < \infty$ and $\alpha(a, b) \geq 1$ for all $a, b \in X$ such that a and b are fixed points of T then $D(z, Tz) = 0$.*

Proof. Since z is a fixed point of T, $M(z,z) = D(z,Tz) = D(z,z) = D(Tz,Tz) < \infty$. Suppose $D(z,Tz) > 0$. Using the contraction, we have

$$\begin{aligned}
\psi(D(z,Tz)) &= \psi(D(Tz,Tz)) \\
&\leq \alpha(z,z)\psi(D(Tz,Tz)) \\
&\leq \beta(M(z,z))\psi(D(M(z,z))) \\
&< \psi(D(z,Tz))
\end{aligned}$$

a contradiction. Thus $D(z,Tz) = 0$.

□

The following result present the uniqueness of fixed point.

Theorem 3.1.3. *From Theorem 3.1.1, let z, z' be fixed points of T. If $\alpha(a,b) \geq 1$ for all $a, b \in X$ such that a and b are fixed points of T and $D(z,Tz), D(z',Tz')$ and $D(z,z')$ are finite, then $z = z'$.*

Proof. To show that $D(z,z') = 0$. Suppose $D(z,z') > 0$. By Lemma 3.1.2, $D(z,Tz) = D(z',Tz') = 0$. Consider

$$\begin{aligned}
\psi(D(z,z')) &= \psi(D(Tz,Tz')) \\
&\leq \alpha(z,z')\psi(D(Tz,Tz')) \\
&\leq \beta(\psi(M(z,z'))) \cdot \psi(M(z,z')) \\
&< \psi(M(z,z'))
\end{aligned}$$

where $M(z,z') = \max\{D(z,z'), D(z,Tz), D(z',Tz')\} = D(z,z')$.

This implies that $\psi(D(z,z')) < \psi(D(z,z'))$ which is a contradiction. Thus $D(z,z') = 0$ implies $z = z'$. □

The result from Theorem 3.1.1 has many consequences.

Corollary 3.1.4. *Let (X, D) be a complete RS-generalized metric space, $\alpha : X \times X \to \mathbb{R}$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold*

1. *there exists $\beta \in \mathcal{F}$ such that for all x, y*

$$\alpha(x,y)D(Tx,Ty) \leq \beta(M(x,y))M(x,y)$$

where $M(x,y) = \max\{D(x,y), D(x,Tx), D(y,Ty)\}$,

2. *T is triangular α-admissible,*

3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(x_0, x_0) \geq 1$ and $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*

4. *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Proof. Let $\psi(t) = t$ in Theorem 3.1.1 and obtain this result immediately. □

Corollary 3.1.5. *Let (X, D) be a complete RS-generalized metric space, $\alpha : X \times X \to \mathbb{R}$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold*

1. *there exists $\beta \in \mathcal{F}$ such that for all x, y*

$$\alpha(x,y)\psi(D(Tx,Ty)) \leq \beta(\psi(D(x,y)))\psi((D(x,y))),$$

2. *T is triangular α-admissible,*

3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(x_0, x_0) \geq 1$ and $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*

4. *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Proof. Follow the proof in Theorem 3.1.1 and obtain this corollary instantly. □

Corollary 3.1.6. *Let (X, D) be a complete RS-generalized metric space, $\alpha : X \times X \to \mathbb{R}$ and $T : X \to X$ be a mapping. Suppose that the following conditions hold*

1. *there exists $\beta \in \mathcal{F}$ such that for all x, y*

$$\alpha(x,y)D(Tx,Ty) \leq \beta(D(x,y))D(x,y),$$

2. *T is triangular α-admissible,*

3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, $\alpha(x_0, x_0) \geq 1$ and $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*

4. *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Proof. Let $\psi(t) = t$ in Corollary 3.1.5 and obtain this result instantly. □

Corollary 3.1.7. *Let (X, D) be a complete RS-generalized metric space and $T : X \to X$ be a mapping. Suppose that the following conditions hold*

1. *there exists $\beta \in \mathcal{F}$ such that for all x, y*

$$D(Tx, Ty) \leq \beta(D(x, y))D(x, y),$$

2. *there exists $x_0 \in X$ such that $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*

3. *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_0\}$ converges to z.

Proof. Let $\alpha(x, y) = 1$ for all $x, y \in X$ in Corollary 3.1.6 and obtain the result directly. □

Remark 3.1.8. 1. *In [1], the authors proved Corollary 3.1.4 in the setting of metric space.*

2. *In [12], the author proved Corollary 3.1.5 in the setting of metric space.*

3. *In [1], the authors proved Corollary 3.1.6 in the setting of metric space.*

4. *In [8], the author proved Corollary 3.1.7 in the setting of metric space.*

3.2 JKS Contraction

Since the RS-generalized metric space can support the value of infinity, we modify the JKS contraction for its compatibility on RS-generalized space.

Definition 3.2.1. Let Θ' be the set of all functions $\theta : (0, \infty] \to (1, \infty]$ satisfying the following conditions

($\theta'1$) θ is nondecreasing,

($\theta'2$) for each sequence $\{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \iff \lim_{n\to\infty} t_n = 0,$$

($\theta'3$) θ is continuous.

Remark 3.2.2. *Since θ is nondecreasing, we have $\theta(\infty) = \infty$. The example of $\theta \in \Theta$ from Example 2.4.2 is still usable in this case.*

The second main result in this thesis is shown below.

Theorem 3.2.3. *Let (X, D) be a complete RS-generalized metric space and $T : X \to X$ be a mapping. Suppose that these conditions hold*

1. *T is a modified JKS contraction, that is, there exist $\theta \in \Theta'$ and $k \in (0, 1)$ such that*

$$x, y \in X, D(Tx, Ty) \neq 0 \Rightarrow \theta(D(Tx, Ty)) \leq (\theta(M(x, y)))^k$$

where $M(x, y) = \max\{D(x, y), D(x, Tx), D(y, Ty)\}$,

2. *there exists $x_0 \in X$ such that $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*

then the Picard sequence $\{x_n\}$ converges to some point $x^* \in X$. Moreover, if $D(x^*, Tx^*) < \infty$, then x^* is a fixed point of T. Moreover, if x', x^* are the fixed points of T and $D(x', x^*)$ is finite, then $x' = x^*$.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$.. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N}$, that x_{n^*} is a fixed point. Thus, we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By Proposition 2.5.10 a Picard sequence must be infinite or almost periodic.

Suppose that $\{x_n\}$ is almost periodic. Then, there exists $n' \geq n_0$ such that $\{x_n\}_{n \geq n^*} = \{x_{n'}, x_{n'+1}, ..., x_{n'+q}\}$ for some $q \in \mathbb{N}$. Consider

$$\begin{aligned} \theta(D(x_{n'+2q}, x_{n'+2q+1})) &= \theta(D(Tx_{n'+2q-1}), Tx_{n'+2q}) \\ &\leq [\theta(M(x_{n'+2q-1}, x_{n'+2q}))]^k \\ &< \theta(M(x_{n'+2q-1}, x_{n'+2q})), \end{aligned}$$

where

$$\begin{aligned} M(x_{n'+2q-1}, x_{n'+2q}) &= \max\{D(x_{n'+2q-1}, x_{n'+2q}), D(x_{n'+2q-1}, Tx_{n'+2q-1}), D(x_{n'+2q}, Tx_{n'+2q})\} \\ &= \max\{D(x_{n'+2q-1}, x_{n'+2q}), D(x_{n'+2q}, x_{n'+2q+1})\}. \end{aligned}$$

If $M(x_{n'+2q-1}, x_{n'+2q}) = D(x_{n'+2q}, x_{n'+2q+1})$, we have

$$\theta(D(x_{n'+2q}, x_{n'+2q+1})) < \theta(M(x_{n'+2q-1}, x_{n'+2q})) = \theta(D(x_{n'+2q}, x_{n'+2q+1}))$$

as a contradiction. Thus,

$$M(x_{n'+2q-1}, x_{n'+2q}) = D(x_{n'+2q-1}, x_{n'+2q})$$

and

$$\theta(D(x_{n'+2q}, x_{n'+2q+1})) \leq [\theta(D(x_{n'+2q-1}, x_{n'+2q}))]^k.$$

Repeat this argument for q times, we have

$$\begin{aligned}\theta(D(x_{n'+2q}, x_{n'+2q+1})) &\leq [\theta(D(x_{n'+2q-1}, x_{n'+2q}))]^k \\ &\leq [\theta(D(x_{n'+2q-2}, x_{n'+2q-1}))]^{k^2} \\ &\leq \dots \\ &\leq [\theta(D(x_{n'+q}, x_{n'+q+1}))]^{k^q} \\ &= [\theta(D(x_{n'+2q}, x_{n'+2q+1}))]^{k^q} \\ &< \theta(D(x_{n'+2q}, x_{n'+2q+1}))\end{aligned}$$

as a contradiction. Therefore, a sequence $\{x_n\}$ is infinite.

Next, we will show that $D(x_{n+1}, x_n) \to 0$ as $n \to \infty$. For any $n \geq n_0$, consider

$$\begin{aligned}\theta(D(x_{n+1}, x_{n+2})) &= \theta(D(Tx_n, Tx_{n+1})) \\ &\leq [\theta(M(x_n, x_{n+1}))]^k \\ &< \theta(M(x_n, x_{n+1})),\end{aligned}$$

where

$$\begin{aligned}M(x_n, x_{n+1}) &= \max\{D(x_n, x_{n+1}), D(x_{n+1}, Tx_{n+1}), D(x_n, Tx_n)\} \\ &= \max\{D(x_n, x_{n+1}), D(x_{n+1}, x_{n+2})\}.\end{aligned}$$

If $M(x_n, x_{n+1}) = D(x_{n+1}, x_{n+2})$, we have

$$\theta(D(x_{n+1}, x_{n+2})) < \theta(M(x_n, x_{n+1})) = \theta(D(x_{n+1}, x_{n+2}))$$

as a contradiction. Thus, $M(x_n, x_{n+1}) = D(x_n, x_{n+1})$ for $n \geq n_0$.

Therefore, for any $n \geq n_0$,

$$\theta(D(x_{n+1}, x_{n+2})) < \theta(D(x_n, x_{n+1})).$$

Since θ is nondecreasing, if $D(x_{n+1}, x_{n+2}) > D(x_n, x_{n+1})$, then $\theta(D(x_{n+1}, x_{n+2})) \geq \theta(D(x_n, x_{n+1}))$ which is a contradiction. Therefore

$$D(x_{n+1}, x_{n+2}) \leq D(x_n, x_{n+1}) \text{ for } n \geq n_0.$$

Thus, $\{D(x_{n+1}, x_n)\}_{n \geq n_0}$ is a nonincreasing sequence. Since D is nonnegative, the limit of $\{D(x_{n+1}, x_n)\}_{n \geq n_0}$ exists, we can suppose that $\lim_{n\to\infty} D(x_{n+1}, x_n) = \epsilon > 0$. Consider when $p > n_0$,

$$\theta(D(x_p, x_{p-1})) = \theta(D(Tx_{p-1}, Tx_{p-2}))$$

$$
\begin{aligned}
&\leq \theta(M(x_{p-1}, x_{p-2}))^k \\
&< \theta(M(x_{p-1}, x_{p-2}))
\end{aligned}
$$

where

$$
\begin{aligned}
M(x_{p-1}, x_{p-2}) &= \max\{D(x_{p-1}, x_{p-2}), D(x_{p-1}, Tx_{p-1}), D(x_{p-2}, Tx_{p-2})\} \\
&= \max\{D(x_{p-1}, x_{p-2}), D(x_p, x_{p-1})\}.
\end{aligned}
$$

If $M(x_{p-1}, x_{p-2}) = D(x_p, x_{p-1})$, we reach a contradiction since $\theta(D(x_p, x_{p-1})) < \theta(D(x_p, x_{p-1}))$. Therefore, $M(x_{p-1}, x_{p-2}) = D(x_{p-1}, x_{p-2})$ for $p > n_0$. Repeating this argument until we reach x_{n_0} and we have

$$
\begin{aligned}
1 &\leq \theta(D(x_p, x_{p-1})) \\
&= \theta(D(Tx_{p-1}, Tx_{p-2})) \\
&\leq \theta(D(x_{p-1}, x_{p-2}))^k \\
&\leq \theta(D(x_{p-2}, x_{p-3}))^{k^2} \\
&\leq \dots \\
&\leq \theta(D(x_{n_0+1}, x_{n_0}))^{k^{p-n_0-1}} \\
&\leq \theta(\delta_{n_0}(D, T, x_0))^{k^{p-n_0-1}}.
\end{aligned}
$$

Taking limit over p both sides, we have

$$
1 \leq \lim_{p\to\infty} \theta(D(x_p, x_{p-1})) \leq \lim_{p\to\infty} \theta(\delta_{n_0}(D, T, x_0))^{k^{p-n_0-1}} = 1.
$$

Thus, we have $\lim_{p\to\infty} \theta(D(x_p, x_{p-1})) = 1$, using the property of contraction and we obtain that $\lim_{p\to\infty} D(x_p, x_{p-1}) = 0$ as a contradiction. Thus $\lim_{n\to\infty} D(x_{n+1}, x_n) = 0$.

Now, we will show that $\{x_n\}$ is a D-Cauchy sequence. Suppose not, then there exist $\epsilon > 0$ and subsequences $\{x_{m_j}\}, \{x_{n_j}\}$ such that $m_j \geq n_j \geq j$ and m_j is the smallest index for which $D(x_{m_j}, x_{n_j}) \geq \epsilon$.

Consider when $j \geq n_0$, we have

$$
\begin{aligned}
\theta(\epsilon) &\leq \theta(D(x_{m_j}, x_{n_j})) \\
&\leq [\theta(M(x_{m_j-1}, x_{n_j-1}))]^k \\
&< \theta(M(x_{m_j-1}, x_{n_j-1}))
\end{aligned}
$$

where

$$
M(x_{n_j-1}, x_{m_j-1}) = \max\{D(x_{n_j-1}, x_{m_j-1}), D(x_{n_j-1}, Tx_{n_j-1}), D(x_{m_j-1}, Tx_{m_j-1})\}
$$

$$= \max\{D(x_{n_j-1}, x_{m_j-1}), D(x_{n_j-1}, x_{n_j}), D(x_{m_j-1}, x_{m_j})\}.$$

If $M(x_{m_j-1}, x_{n_j-1}) \neq D(x_{m_j-1}, x_{n_j-1})$, from θ is nondecreasing, we have

$$\theta(\epsilon) < \theta(M(x_{m_j-1}, x_{n_j-1})) \Rightarrow \epsilon < M(x_{m_j-1}, x_{n_j-1}).$$

Taking limit over j, we obtain

$$\epsilon = \lim_{j\to\infty} \epsilon \leq \lim_{j\to\infty} M(x_{m_j-1}, x_{n_j-1}) = 0.$$

as a contradiction. Thus, $M(x_{m_j-1}, x_{n_j-1}) = D(x_{m_j-1}, x_{n_j-1})$ for all $j \geq n_0$. We repeat this method and get

$$\begin{aligned}
\theta(\epsilon) &\leq \theta(D(x_{m_j}, x_{n_j})) \\
&\leq [\theta(D(x_{m_j-1}, x_{n_j-1}))]^k \\
&\leq [\theta(D(x_{m_j-2}, x_{n_j-2}))]^{k^2} \\
&\leq \dots \\
&\leq [\theta(M(x_{m_j-(n_j-n_0)}, x_{n_0}))]^{k^{n_j-n_0-1}} \\
&\leq [\theta(\delta_{n_0}(D, T, x_0))]^{k^{n_j-n_0-1}}.
\end{aligned}$$

Taking limit over j, we have

$$\theta(\epsilon) \leq \lim_{j\to\infty} [\theta(\delta_{n_0}(D, T, x_0))]^{k^{n_j-n_0-1}} = 1$$

as a contradiction. Thus, $\{x_n\}$ must be a D-Cauchy sequence. Since X is complete, $\{x_n\}$ must converges to some $x^* \in X$.

Assume that $D(x^*, Tx^*)$ is finite. We will prove that $x^* = Tx^*$. Suppose $D(x^*, Tx^*) > 0$. First, we show that there exists $M' \in \mathbb{N}$ such that $D(x_n, Tx^*) > 0$ for all $n > M'$. We consider in 2 cases:

1. if $Tx^* \notin O_T(x_0)$, we have $Tx^* \neq T^n x_0$ for all $n \in \mathbb{N}$ which implies that for all $M' \in \mathbb{N}$, $D(x_n, Tx^*) > 0$ for all $n > M'$.

2. if $Tx^* \in O_T(x_0)$, then there exists $m \in \mathbb{N}$ such that $Tx^* = T^m x_0$. Since $\{x_n\}$ is infinite, $Tx^* \notin \{x_n\}_{n>m}$. We can choose any M' such that $M' > m$.

Let $M = \max\{n_0, M' + 1\}$. For $n > M$ we have

$$\theta(D(T^{n+1}x_0, Tx^*)) \leq [\theta(M(T^n x_0, x^*))]^k$$

where

$$M(T^n x_0, x^*) = \max\{D(T^n x_0, x^*), D(T^n x_0, T^{n+1} x_0), D(x^*, Tx^*)\}.$$

We can easily see that

$$\lim_{n\to\infty} M(T^n x_0, x^*) = D(x^*, Tx^*). \tag{3.1}$$

Now, we take limit over n both side and obtain

$$\lim_{n\to\infty} \theta(D(T^{n+1}x_0, Tx^*)) \leq \lim_{n\to\infty} [\theta(M(T^n x_0, x^*))]^k.$$

This implies

$$\begin{aligned}
\theta(D(x^*, Tx^*)) &= \theta(\lim_{n\to\infty} D(T^{n+1}x_0, Tx^*)) \\
&= \lim_{n\to\infty} \theta(D(T^{n+1}x_0, Tx^*)) \\
&\leq \lim_{n\to\infty} [\theta(M(T^n x_0, x^*))]^k \\
&= [\theta(\lim_{n\to\infty} M(T^n x_0, x^*))]^k \\
&= [\theta(D(x^*, Tx^*))]^k \\
&< \theta(D(x^*, Tx^*))
\end{aligned}$$

as a contradiction. Thus, $D(x^*, Tx^*) = 0$ and x^* is a fixed point as required.

Suppose x' to be another fixed point of T and $D(x', x^*) > 0$ but finite, we have

$$\begin{aligned}
\theta(D(x', x^*)) &= \theta(D(Tx', Tx^*)) \\
&\leq [\theta(D(x', x^*))]^k \\
&< \theta(D(x', x^*))
\end{aligned}$$

as a contradiction. Thus, $x' = x^*$. □

This result has many consequences.

Corollary 3.2.4. *Let (X, D) be a complete JS-generalized metric space and $T : X \to X$ be a mapping. Suppose that these conditions hold*

1. *T is a modified JKS contraction,*
2. *there exists $x_0 \in X$ such that $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*

then the Picard sequence $x_n = T^n x_0$ converges to some point $x^* \in X$. If $D(x^*, Tx*) < \infty$, then x^* is a fixed point of T. Moreover, if x, x^* are the fixed points of T and $D(x, x^*)$ is finite, then $x = x^*$.

Corollary 3.2.5. *Let (X, D) be a complete dislocated metric space and $T : X \to X$ be a mapping. Suppose that these conditions hold*

1. *T is a modified JKS contraction,*
2. *there exists $x_0 \in X$ such that $\delta_{n_0}(D, T, x_0)$ is finite for some $n_0 \in \mathbb{N}$,*

then the Picard sequence $x_n = T^n x_0$ converges to some point $x^* \in X$ and x^* is a fixed point of T. Moreover, if x, x^* are the fixed points of T, then $x = x^*$.

Corollary 3.2.6. *Let (X, D) be a complete RS-generalized metric space and $T : X \to X$ be a mapping. Suppose that there exists $\lambda \in (0, 1)$ such that*

$$D(Tx, Ty) \leq \lambda \max\{D(x, y), D(x, Tx), D(y, Ty)\}.$$

Then the Picard sequence $x_n = T^n x_0$ converges to some point $x^* \in X$. If $D(x^*, Tx*) < \infty$, then x^* is a fixed point of T. Moreover, if x, x^* are the fixed points of T and $D(x, x^*)$ is finite, then $x = x^*$.

Proof. From the given contraction, we have

$$e^{D(Tx,Ty)^{\frac{1}{2}}} \leq [e^{\max\{D(x,y),D(x,Tx),D(y,Ty)\}^{\frac{1}{2}}}]^{\lambda^{\frac{1}{2}}}.$$

Since $\theta(t) = e^{t^{\frac{1}{2}}}$ satisfies all the conditions for Theorem 3.2.3, we can conclude as in Theorem 3.2.3. □

Example 3.2.7. Let (Y, d) be as Example 2.3.2. We can easily see that $\lim_{n\to\infty} d(\frac{1}{n}, 0) = \lim_{n\to\infty} d(\frac{1}{n}, 2) = 0$. Therefore, (Y, d) is a B_2-space contain a sequence that D-converges to two different points. Let $Z = \{3, 4, 5, 6\}$. Now, let $X = Y \cup Z$ and define $D : X \times X \to [0, \infty]$ as Example 2.5.14. We can see that (X, D) is a complete RS-generalized space.

$$T(x) = \begin{cases} 0, & \text{if } x \in Y, \\ 5, & \text{if } x \in \{5, 6\}, \\ 6, & \text{if } x \in \{3, 4.\} \end{cases}$$

In this case, we let $M = 4$.

For this mapping, there exist $\theta(t) = e^{t^{\frac{1}{2}}}$ for all $t \in (0, \infty]$ and $k \in (\sqrt{\frac{2}{3}}, 1)$ that satisfies the condition.

Consider when $D(Tx, Ty) \neq 0$. It's easy to see that there are two cases available.

1. If $D(x, y) = \infty$, then $Tx \in Y$ and $Ty \in Z$ or vice versa which implies $M(x, y) = D(x, y) = \infty$ and

$$\theta(D(Tx,Ty)) = \theta(\infty) = \infty = [\theta(\infty)]^k = \theta(M(x,y))^k.$$

Before we consider another case, let we observe the contraction itself by:

$$\theta(D(Tx,Ty)) = e^{D(Tx,Ty)^{\frac{1}{2}}} \leq [e^{M(x,y)^{\frac{1}{2}}}]^k = \theta(M(x,y))^k = e^{kM(x,y)^{\frac{1}{2}}}.$$

Take log over both sides, we have

$$D(Tx,Ty)^{\frac{1}{2}} \leq kM(x,y)^{\frac{1}{2}} \Rightarrow D(Tx,Ty) \leq k^2 M(x,y).$$

Thus, we can consider this equation instead of the contraction itself.

2. If $D(x,y) < \infty$, we have $Tx \neq Ty$ $Tx, Ty = \{5,6\}$. There are 2 possible cases here.

   - If $x = 3$, then $y \in \{5,6\}$. We have $M(3,5) = \max\{D(3,5), D(3,6), D(5,5)\} = \max\{2,4,0\} = 4$ and $M(3,6) = \max\{D(3,6), D(3,6), D(6,5)\} = \max\{2,4\} = 4$, then

     $$\begin{aligned} 2 \leq k^2(4) &\Rightarrow \frac{2}{4} \leq k^2 \\ &\Rightarrow \sqrt{\frac{2}{4}} \leq k. \end{aligned}$$

     Since $\sqrt{\frac{2}{4}} < \sqrt{\frac{2}{3}} < k$, our k is usable.

   - If $x = 4$, then $y \in \{5,6\}$. We have $M(4,6) = M(4,5) = \max\{D(4,5), D(4,6), D(5,6)\} = \max\{2,3\} = 3$ and

     $$\begin{aligned} 2 \leq k^2(3) &\Rightarrow \frac{2}{3} \leq k^2 \\ &\Rightarrow \sqrt{\frac{2}{3}} \leq k. \end{aligned}$$

There exists $0 \in X$ such that $\delta(D,T,0) < \infty$. Now all condition in Theorem 3.2.3 hold. Thus, there exist fixed points of T, in this case, they are 0 and 5.