CHAPTER 1

Introduction

Algebraic graph theory is a fascinating branch of mathematics in which algebraic methods are accordingly applied to problems about graphs. It is a well-known subject that is eventually a combination of algebra and graph theory. In algebraic graph theory, mathematicians study how algebraic methods are utilized in order to provide surprising results in graph theory. Algebraic tools can be used to give elegant proofs of graph theoretic facts, and there are many interesting algebraic objects associated with graphs. One of the main topics of algebraic graph theory involves the study of graphs in connection to group theory. The interesting connection with group theory is that, given any group, corresponding graphs known as Cayley graphs can be constructed, and these graphs have properties related to the structure of the group. Many mathematicians extended their viewpoints to construct Cayley digraphs of semigroups and studied further corresponding results which are more general than those results from the group case.

Given a semigroup S and a subset A of S. The Cayley digraph $\mathrm{Cay}(S,A)$ of a semigroup S with a connection set A is defined to be a digraph with a vertex set $V(\mathrm{Cay}(S,A)) = S$ and an arc set $E(\mathrm{Cay}(S,A)) = \{(x,xa) : x \in S \text{ and } a \in A\}$. It is clear that if we take A to be an empty subset of S, then $\mathrm{Cay}(S,A)$ is considered to be an empty digraph. So in this work, we consequently need to focus on a nonempty connection set.

The Cayley graph was first considered for finite groups by Arthur Cayley (1821 − 1895), British mathematician, in 1878. It is a graph that encodes the abstract structures of a group. Many new interesting results of Cayley graphs of groups have been investigated and popularly studied by various authors. Furthermore, Cayley digraphs of semigroups are also focused to study more results which are interesting topics in the field of algebraic graph theory. They have been extensively studied in several journals. One of the appealing ways in the study of Cayley graphs of semigroups is considering how to apply the results obtained from the Cayley graphs of groups to the case of semigroups. A rectangular group is one of the popular semigroups that has been widely considered in the construction of Cayley digraphs which can be seen in further references.

There are numerous mathematical researchers who considered Cayley graphs or digraphs of various semigroups in recent years. In 2002, Kelarev ([30]) presented some results on undirected Cayley graphs. In 2003, Kelarev and Praeger ([31]) studied the transitivity of Cayley graphs of groups and semigroups. In that year, Kelarev and Quinn ([32]) obtained the combinatorial property for Cayley graphs of semigroups. In 2004, Panma, Knauer and Arworn ([46]) studied the transitive Cayley graphs of right (left) groups and of Clifford semigroups. Later in 2006, Panma, Na Chiangmai, Knauer and Arworn ([49]) presented some characterizations of Clifford semigroup digraphs. In 2007, Panma ([44]) generally provided the characterizations of Cayley graphs of some completely simple semigroups. In 2009, Khosravi ([33]) studied some properties of Cayley graphs of left groups. In the same year, Panma, Knauer and Arworn ([47]) verified the results for transitive Cayley graphs of strong semilattice of right (left) groups. Later in 2010, Hao and Luo ([24]) obtained more results about Cayley graphs of left groups and right groups. In the same year, Khosravi and Mahmoudi ([35]) and Panma ([45]) studied some properties and indicated the characterizations of Cayley graphs of rectangular groups with arbitrary connection sets, respectively. In 2012, Khosravi and Khosravi ([34]) gave the characterizations of Cayley graphs of Brandt semigroups.

Studying on several graph parameters is an another interesting topic in algebraic graph theory consisting of the domination parameters and the independence numbers of graphs or digraphs. The domination problem is one of the most interesting concepts which was studied from the 1950s onwards, but the average rate of researches on domination parameters increased dramatically in the mid$-$1970s. Some results about the domination parameters for Cayley graphs have been considered by many mathematical researchers. There are several types of domination parameters studied on special classes of graphs and digraphs such as the domination number, total domination number, and independent domination number.

In 1998, Haynes, Hedetniemi and Slater ([27]) described the domination number of various graphs. In 2007, Chelvam and Rani ([12]) studied the dominating sets of Cayley graphs on $\mathbb{Z}_n$. Later in 2009, Rad ([50]) obtained the domination number of circulant graphs. Moreover, Kostochka and Stodolsky ([38]) presented an upper bound for the domination number of $n-$vertex connected cubic graphs. They showed that the domination number of those graphs is less than or equal to $4n/11$ where $n > 8$. In the same year, Kostochka and Stocker ([37]) improved the upper bound obtained by Stodolsky, they proved that the domination number of every $n-$vertex cubic connected graph is at most $\lfloor 5n/14 \rfloor$ for $n > 8$. In 2010, Huang and Xu ([29]) presented the domination number of a path P_n and a cycle C_n on n vertices. In the next year, Murugesan and Nair ([43]) proved that the domination number of a cubic bipartite graph is less than or equal to 1/3 of the number of vertices. In 2012, Chelvam and Mutharasu ([11]) obtained the bounds for domination parameters in circulant graphs. In that year, Chelvam and Kalaimurugan ([9]) verified that the domination number of a $k-$regular graph of n vertices is greater than or equal to $n/(k+1)$.

In 1980, Cockayne, Dawes and Hedetniemi ([17]) described some results of the total domination number in graphs. In 1995, Sun ([53]) showed that the total domination number of a connected graph of order n with a minimum degree $\delta \geq 2$ is less than or equal to $\lfloor 4(n+1)/7 \rfloor$. In 2007, Arumugam, Jacob and Volkmann ([3]) presented the bounds for the total domination number of some digraphs which depend on the maximum out–degree. Two years later, Chelvam ([8]) indicated the total domination number in circulant graphs. In 2010, Chelvam and Rani ([14]) presented some results about the total domination number of Cayley graphs on $\mathbb{Z}_n$. In 2011, Chelvam and Mutharasu ([10]) showed more results of the total domination number in circulant graphs. In the next year, Amos ([2]) investigated a lower bound of the total domination number of some graphs.

In 1991, Haviland ([25]) obtained an upper bound of the independent domination number of some specific graphs related to their minimum and maximum degrees. Moreover, in 1995, Haviland ([26]) also presented some results for the independent domination number of connected regular graphs. In 1998, Glebov and Kostochka ([21]) investigated the independent domination number of graphs with given minimum degree. In 1999, Sun and Wang ([54]) studied the independent domination number of graphs in terms of the minimum degree. In that year, Lam, Shiu and Sun ([39]) proved that for any connected cubic graph of order $n \geq 8$, its independent domination number is less than or equal to $2n/5$. In 2005, Chen, Ma, Xing and Sun ([15]) made a note on the independent domination number of subset graphs. In the next year, Duckworth and Wormald ([19]) determined the independent domination number of random regular graphs. In 2009, Chelvam and Rani ([13]) investigated the independent domination number of Cayley graphs of $\mathbb{Z}_n$. In 2012, Goddard, Henning, Lyle and Southey ([23]) obtained the result that if G is an $r-$regular graph of n vertices with $2n/5 \leq r < n/2$, then the independent domination number of G is less than or equal to $(2/3)(n-r)$. In the next year, Goddard and Henning ([22]) presented the bound for the independent domination number of bipartite graphs. They indicated that the independent domination number of any bipartite graph of n vertices without isolated vertices is less than or equal to $n/2$.

In addition, the independence number of several types of graphs or digraphs are spaciously investigated.

In 1990, Frieze ([20]) studied the independence number of random graphs. Later in 1993, Selkow ([51]) obtained the independence number of graphs in terms of degrees. In 1998, Csizmadia ([18]) showed some results about the independence number of minimum distance graphs. In 2005, Lichiardopol ([42]) determined the independence number of iterated line digraphs. Later in 2011, Abay - Asmerom, Hammack, Larson and Taylor ([1]) studied the independence number in the cartesian product of graphs.

Moreover, the structures of morphisms, such as homomorphisms and isomorphisms, on Cayley digraphs of semigroups are also interesting to study because many researchers have obtained some results of Cayley digraphs of semigroups by using the properties of their homomorphisms.

In 1994, Bauslaugh ([4]) studied various properties for homomorphisms of the infinite directed graphs. In 1998, Li ([40]) considered the isomorphisms of connected Cayley graphs. In 2002, Li ([41]) also presented more results on isomorphisms of finite Cayley graphs. In 2006, Cameron ([6]) defined an equivalence relation on the family of graphs via the graph homomorphisms. In 2012, Ruangnai, Panma and Arworn showed some results on Cayley isomorphisms of left and right groups. In 2013, Shurbert ([52]) studied some parameters of graphs by considering their homomorphisms. Recently in 2016, Panma and Meksawang ([48]) presented the isomorphism conditions for Cayley graphs of rectangular groups.

Hereafter, we study some properties on Cayley digraphs of finite rectangular groups related to their connection sets. The purpose of this thesis is to study some well-known and interesting concepts in Cayley digraphs of rectangular groups. In this thesis, we provide some prominent results of the domination number, total domination number, independent domination number, independence parameters, and some characterizations of endomorphisms on Cayley digraphs of rectangular groups. All sets mentioned in this thesis are considered to be finite sets.

This thesis consists of 8 chapters which are organized as follows.

The first chapter is described to present some backgrounds for the literature review of related researches.

In Chapter 2, we present some valuable basic preliminaries such as the definitions, notations, examples and some known results which are useful for the sequel. We also refer to further references for more information which is not mentioned in this thesis.

In Chapter 3, certain characterizations of Cayley digraphs of rectangular groups relative to their connection sets are investigated. Moreover, some basic properties of Cayley digraphs of rectangular groups are also presented.

In Chapter 4, we show our main results of this work. The domination parameters of Cayley digraphs of rectangular groups comprising of the domination number and the total domination number are presented. Some corresponding examples are illustrated to guarantee our results, certainly.

In Chapter 5, further results of the independence number, weakly independence number, dipath independence number, and weakly dipath independence number of Cayley digraphs of rectangular groups with their connection sets are provided.

In Chapter 6, some consequential results of independent domination parameters such as the independent domination number, weakly independent domination number, dipath independent domination number, and weakly dipath independent domination number are completely indicated.

In Chapter 7, we describe some prominent characterizations of endomorphisms on Cayley digraphs of rectangular groups with their connection sets.

The last chapter summarizes the information of the whole work. It also provides some important results of this thesis.