CHAPTER 2

Preliminaries

This chapter provides the basic definitions, terminologies and notations from further references which are useful for this thesis. We divide this chapter into 6 parts consisting of semigroups, digraphs, Cayley digraphs of semigroups, domination parameters, independence parameters, and digraph morphisms. We emphasize that all sets appeared in this thesis are considered to be finite and we also denote by $|A|$ the cardinality of the set A. Now, we are ready to describe more details of those parts.

2.1 Semigroups

This part gives more information about semigroups taken from [16] and [28].

Definition 2.1.1. Let S be a nonempty set. A *binary operation* on S is a function $\cdot$ from $S \times S$ into S. The image of (a, b) in S will be denoted by $a \cdot b$. For convenience, we shall omit the dot, writing ab for $a \cdot b$. A *semigroup* is a nonempty set S together with a binary operation on S that satisfies the associative property, that is, $a(bc) = (ab)c$ for all $a, b, c \in S$.

An element e of a semigroup S is called an *identity element* of S if $es = s = se$ for all $s \in S$. An element $u \in S$ is called a *unit* if there exists $b \in S$ such that $ub = e = bu$ where e is an identity element of S. A semigroup which has an identity element is called a *monoid*. A monoid S is said to be a *group* if s is a unit for all $s \in S$.

Definition 2.1.2. A nonempty subset T of a semigroup S is called a *subsemigroup* of S if it is closed under the binary operation of S, that is, for all $x, y \in T$, $xy \in T$. Moreover, a subsemigroup of S which is a group with respect to the binary operation inherited from S will be called a *subgroup* of S.

Definition 2.1.3. Let S be a semigroup and A a nonempty subset of S. The *subsemigroup of S generated by A*, denoted by $\langle A \rangle$, is the semigroup consisting of all elements in S that can be expressed as finite products of elements in A, that is,

$$\langle A \rangle = \{a_1 a_2 a_3 \cdots a_n : a_i \in A \text{ where } 1 \leq i \leq n \text{ and } n \in \mathbb{N}\}.$$

Furthermore, if S is a group, then the *subgroup of S generated by A* is $\langle A \cup A^{-1} \rangle$ where $A^{-1} = \{a^{-1} : a \in A\}$.

Definition 2.1.4. Let S be a semigroup. For any nonempty subsets X and Y of S, define

$$XY = \{xy \in S : x \in X \text{ and } y \in Y\}.$$

For any subsemigroup T of S and $a \in S$, define $aT = \{at \in S : t \in T\}$.

In addition, if we let G be a group, H a subgroup of G and g an element in G, then the set gH is called *a left coset of H in G* and the set of all different left cosets of H in G will be denoted by $G/H = \{aH : a \in G\}$. Moreover, the number of different left cosets of H in G is called the *index of H in G* and written as $[G : H]$.

Next, we will prescribe the definition of semigroup morphisms which are useful for defining other special semigroups studied in this thesis.

Definition 2.1.5. Let S and T be semigroups. A mapping ϕ from S to T is called a *homomorphism* if $(ab)\phi = (a\phi)(b\phi)$ for all $a, b \in S$. If a homomorphism ϕ is one-to-one, we call it a *monomorphism*, if it is onto, we call it an *epimorphism*, and if it is both one-to-one and onto, we shall call it an *isomorphism.* If ϕ is a homomorphism from S to S, we call it an *endomorphism* of S. An isomorphism from S to S will be called an *automorphism* of S. If there exists an isomorphism between S and T, we say that S and T are *isomorphic* and write $S \cong T$.

Definition 2.1.6. Let $n \in \mathbb{N}$ and $I = \{1, 2, \ldots, n\}$ be an index set. Let $\{A_i : i \in I\}$ be a family of nonempty sets indexed by I. The *Cartesian product* $A_1 \times A_2 \times \cdots \times A_n$ is identified with the set of all ordered $n-$tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for all $i \in I$.

For each $k \in I$, define a map $p_k : A_1 \times A_2 \times \cdots \times A_n \to A_k$ by $(a_1, a_2, \ldots, a_n) \mapsto a_k$. The map p_k is called a *projection map* of the product onto its kth component.

Definition 2.1.7. Let $n \in \mathbb{N}$ and $I = \{1, 2, \ldots, n\}$ be an index set. Let $\{S_i : i \in I\}$ be a family of semigroups indexed by I. The Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ becomes a semigroup if we define the binary operation as follows:

$$(s_1, s_2, \ldots, s_n)(t_1, t_2, \ldots, t_n) = (s_1t_1, s_2t_2, \ldots, s_nt_n)$$
$$\text{for all } (s_1, s_2, \ldots, s_n), (t_1, t_2, \ldots, t_n) \in S_1 \times S_2 \times \cdots \times S_n.$$

We refer to this semigroup as the *direct product* of $S_1, S_2, \ldots, S_n$.

Definition 2.1.8. A nonempty subset A of a semigroup S is called a *left ideal* if $SA \subseteq A$ and a *right ideal* if $AS \subseteq A$.

Definition 2.1.9. Let S be a semigroup. An element $z \in S$ is said to be *zero* if $zx = z = xz$ for all $x \in S$. A semigroup S without zero is said to be *left (right) simple* if it has no proper left (right) ideals.

Definition 2.1.10. A semigroup S is said to be *left cancellative* if for each $a, b, c \in S$, $ab = ac$ implies $b = c$ and said to be *right cancellative* if for each $a, b, c \in S$, $ac = bc$ implies $a = b$.

We now present the definitions of various special semigroups comprising of a left group, a right group and a rectangular group which are focused in this thesis.

Definition 2.1.11. A semigroup S is said to be a *left (right) group* if S is left (right) simple and right (left) cancellative. In addition, the semigroup S is called a *left (right) zero semigroup* if $xy = x$ ($xy = y$) for all $x, y \in S$.

It is known that a semigroup S is a left (right) group if and only if S is isomorphic to the direct product of a group and a left (right) zero semigroup (see [28]).

Definition 2.1.12. A semigroup S is called a *rectangular group* if S is isomorphic to the direct product of a group, a left zero semigroup and a right zero semigroup.

It is clear that a left (right) zero semigroup and a left (right) group are rectangular groups. The following diagram shows the relationships between a rectangular group, a left (right) group, a left (right) zero semigroup, and a group. An arrow from one property to another one indicates that the former implies the latter.

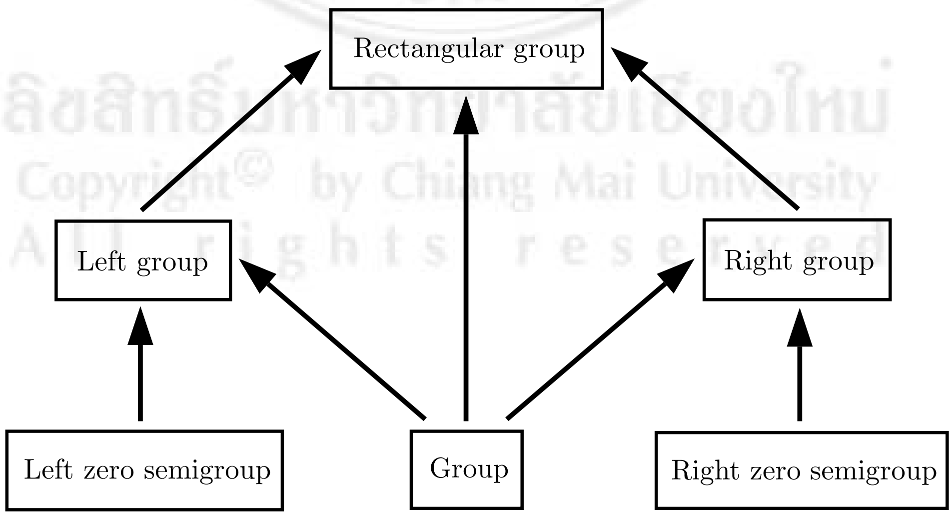

Figure 2.1: Diagram.

2.2 Digraphs

This part presents basic knowledge about digraphs and some algebraic structures of digraphs referred from [5], [7] and [36].

Definition 2.2.1. A *digraph (directed graph)* D is a pair $(V(D), E(D))$ in which $V(D)$ is a nonempty set whose elements are called the *vertices* and $E(D)$ is the subset of the set of ordered pairs of elements in $V(D)$. The elements of $E(D)$ are called the *arcs* of D. The set $V(D)$ is called a *vertex set* of D and the set $E(D)$ is called an *arc set* of D. For each $v \in V(D)$, an arc $(v, v) \in E(D)$ is called a *loop*. The number of elements in the vertex set is called the *order* of D.

Definition 2.2.2. Let D be a digraph without parallel arcs. For each $u, v \in V(D)$ and $e := (u, v) \in E(D)$, we say that:

- u is the *initial vertex (tail)* and v is the *terminal vertex (head)*;
- e is an arc joining from u to v;
- e is *incident* with u and v;
- e is incident from u and e is incident to v;
- u is *adjacent* to v and v is adjacent from u.

For any $v \in V(D)$, the number of arcs incident to v is the *indegree* of v which is denoted by $d^-(v)$. The number of arcs incident from v is called the *outdegree* of v, denoted by $d^+(v)$. Furthermore, we define $N^-(v)$ and $N^+(v)$ by

$$N^-(v) = \{u \in V(D) : (u, v) \in E(D)\} \text{ and } N^+(v) = \{u \in V(D) : (v, u) \in E(D)\}.$$

We note that $d^-(v) = |N^-(v)|$ and $d^+(v) = |N^+(v)|$. The *total degree* (or simply degree) of v is $d(v) = d^-(v) + d^+(v)$. A vertex v for which $d^+(v) = d^-(v) = 0$ is called an *isolated vertex*.

Definition 2.2.3. Let D be a digraph. A digraph H is said to be a *subdigraph* of D if $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. A subdigraph H of D is called a *strong subdigraph* of D if and only if whenever u and v are vertices of H and (u, v) is an arc in D, then (u, v) is an arc in H as well. Furthermore, a subdigraph H is said to be a *spanning subdigraph* of D if $V(H) = V(D)$.

Example 2.2.1. The examples of a digraph, a subdigraph, a strong subdigraph, and a spanning subdigraph are shown as follows.

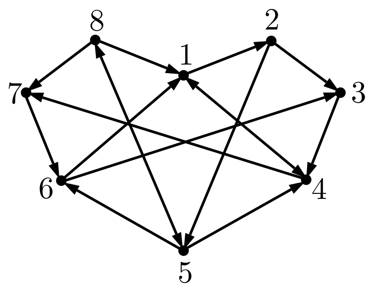

Figure 2.2: Digraph D_1.

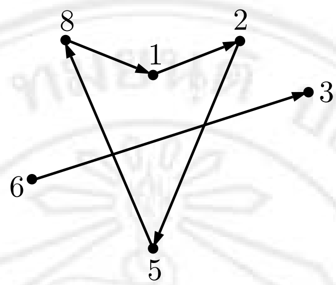

Figure 2.3: Subdigraph of D_1.

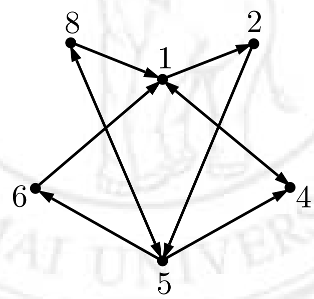

Figure 2.4: Strong subdigraph of D_1.

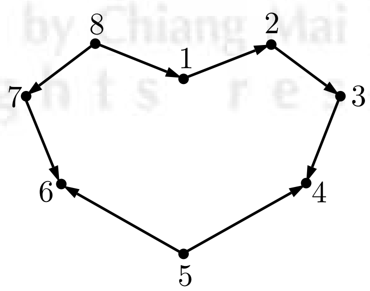

Figure 2.5: Spanning subdigraph of D_1.

Definition 2.2.4. Let D be a digraph and $k \in \mathbb{N}$. A sequence

$$W : u = u_0, u_1, \ldots, u_k = v \tag{2.1}$$

of vertices of D such that u_i is adjacent to u_{i+1} for all i in which $0 \leq i \leq k-1$ is called a $u-v$ *diwalk* ($u-v$ *directed walk*) in D. For $1 \leq i \leq k-1$, each arc (u_i, u_{i+1}) is said to *lie on* or *belong to* W. The number of occurrences of arcs on a diwalk is the *length* of the diwalk. So the length of the diwalk W in (2.1) is k. A diwalk in which no vertex is repeated is called a *dipath (directed path)*. A $u-v$ diwalk is *closed* if $u = v$ and is *open* if $u \neq v$. A closed diwalk of length at least 2 in which no vertex is repeated except for the initial and terminal vertices is a *dicycle (directed cycle)*.

Next, we will consider the special type of graphs which is useful for defining the connectedness concepts of digraphs.

Definition 2.2.5. Let D be a digraph. The *underlying graph* of D is obtained by removing all directions from the arcs of D and replacing any resulting pair of parallel edges by a single edge. Equivalently, the underlying graph of a digraph D is obtained by replacing each arc (u, v) or pair $(u, v), (v, u)$ of arcs by the edge $\{u, v\}$, see [7] for more information.

Example 2.2.2. The following graph is an underlying graph of the digraph D_1 which is shown in Figure 2.2.

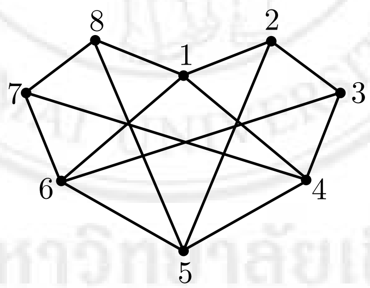

Figure 2.6: Underlying graph of D_1.

The following definition is described for presenting the information about concepts of the connectedness of digraphs. Recall that the connected graph is the graph for which any two different vertices connected by a path of that graph, see [7].

Definition 2.2.6. Let D be a digraph. The digraph D is said to be *connected (weakly connected)* if the underlying graph of D is connected. A maximal connected subdigraph of D is called a *component* of D. Moreover, D is said to be *strongly connected* if D contains both a $u-v$ dipath and a $v-u$ dipath for every pair u, v of distinct vertices of D.

Example 2.2.3. This example illustrates a weakly connected digraph and a strongly connected digraph, respectively.

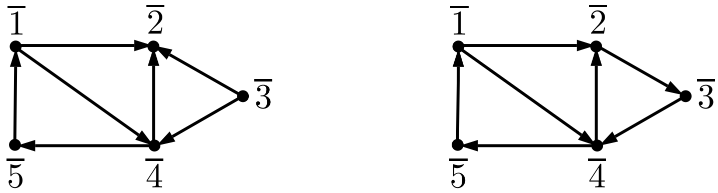

Figure 2.7: Weakly connected digraph & Strongly connected digraph.

Definition 2.2.7. For any family of nonempty sets $\{X_i : i \in I\}$, we write $\dot{\bigcup\limits_{i\in I}} X_i := \bigcup\limits_{i\in I} X_i$ if $X_i \cap X_j = \emptyset$ for all $i \neq j$. Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be digraphs such that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as

$$\dot{\bigcup}_{i=1}^{n}(V_i, E_i) := (\dot{\bigcup}_{i=1}^{n} V_i, \dot{\bigcup}_{i=1}^{n} E_i).$$

Now, we introduce the definition of digraph isomorphisms.

Definition 2.2.8. Let $D_1 := (V_1, E_1)$ and $D_2 := (V_2, E_2)$ be digraphs. A mapping $\varphi : D_1 \to D_2$, in the sense that $\varphi : V_1 \to V_2$, is called a *(digraph) homomorphism* if $(u, v) \in E_1$ implies $(\varphi(u), \varphi(v)) \in E_2$ for all $u, v \in V_1$. Such a mapping φ is said to be *arc-preserving*. A digraph homomorphism $\varphi : D_1 \to D_1$ is called an *(digraph) endomorphism*. We denote by $\mathrm{End}(D_1)$ the set of all endomorphisms of D_1. If φ is a bijection from D_1 to D_2 such that $(u, v) \in E_1$ if and only if $(\varphi(u), \varphi(v)) \in E_2$, then φ is called an *(digraph) isomorphism* and we say that D_1 is *isomorphic* to D_2, denoted by $D_1 \cong D_2$. A digraph isomorphism $\varphi : D_1 \to D_1$ is called an *(digraph) automorphism*.

Next, we will recall the definition of a Cayley digraph of a semigroup with respect to the connection set.

Definition 2.2.9. Let S be a semigroup and A a subset of S. The *Cayley digraph* $\mathrm{Cay}(S, A)$ *of a semigroup* S *with a connection set* A is defined to be a digraph with a vertex set $V(\mathrm{Cay}(S, A)) = S$ and an arc set $E(\mathrm{Cay}(S, A)) = \{(x, xa) : x \in S \text{ and } a \in A\}$.

Example 2.2.4. Let $(\mathbb{Z}_6, +)$ be a group of integers modulo 6 and $A = \{\overline{1}, \overline{2}, \overline{3}\}$ a subset of $\mathbb{Z}_6$. The Cayley digraph of $\mathbb{Z}_6$ with a connection set A is shown as follows.

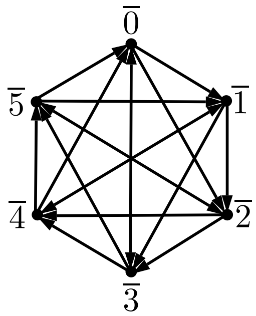

Figure 2.8: $\mathrm{Cay}(\mathbb{Z}_6, \{\overline{1}, \overline{2}, \overline{3}\})$.

Notation 2.2.10. Throughout this thesis, we denote by G a group, L a left zero semigroup, and R a right zero semigroup. For convenience, we shall prescribe some useful notations as follows:

Δ : the Cayley digraph $\mathrm{Cay}(G \times L \times R, A)$ of a rectangular group $G \times L \times R$;

Γ : the Cayley digraph $\mathrm{Cay}(G \times L, A)$ of a left group $G \times L$;

Λ : the Cayley digraph $\mathrm{Cay}(G \times R, A)$ of a right group $G \times R$,

where A is referred as a connection set of such Cayley digraphs. Note that for Cayley digraphs of left groups $G \times L$ and right groups $G \times R$, if $|L| = 1$ or $|R| = 1$, then we can consider such Cayley digraphs as Cayley digraphs of a group G, certainly. So in this thesis, we will consider in the case where $|L| \geq 2$ and $|R| \geq 2$.

Now, we recall some results which are needed in the sequel as below for further references.

Lemma 2.2.5 ([45])**.** *Let $S = G \times L$ be a left group and A a nonempty subset of S. Then the following conditions hold:*

(1). *for each $l \in L$,* $\mathrm{Cay}(G \times \{l\}, p_1(A) \times \{l\}) \cong \mathrm{Cay}(G, p_1(A))$,

(2). $\mathrm{Cay}(S, A) = \dot{\bigcup\limits_{l \in L}} \mathrm{Cay}(G \times \{l\}, p_1(A) \times \{l\})$.

Lemma 2.2.6 ([44])**.** *Let $S = G \times L$ be a left group, A a nonempty subset of S, $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$, and let $I = \{1, 2, \ldots, k\}$. Then*

(1). *$S/\langle A\rangle = \{g_i\langle p_1(A)\rangle \times \{l\} : i \in I, l \in L\}$ and $S = \dot{\bigcup\limits_{i \in I, l \in L}} (g_i\langle p_1(A)\rangle \times \{l\})$,*

(2). $\mathrm{Cay}(S, A) = \dot{\bigcup\limits_{i \in I, l \in L}} ((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ *where $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ is a strong subdigraph of* $\mathrm{Cay}(S, A)$ *with* $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il}) \cong \mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ *for all $i \in I, l \in L$.*

Lemma 2.2.7 ([44])**.** *Let $S = G \times R$ be a right group, A a nonempty subset of S such that $p_2(A) = R, G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$, and let $I = \{1, 2, \ldots, k\}$. Then*

(1). $S/\langle A\rangle = \{g_i\langle p_1(A)\rangle \times R : i \in I\}$ *and* $S = \dot{\bigcup\limits_{i\in I}}(g_i\langle p_1(A)\rangle \times R)$,

(2). $\mathrm{Cay}(S, A) = \dot{\bigcup\limits_{i\in I}}((g_i\langle p_1(A)\rangle \times R), E_i)$ *where* $((g_i\langle p_1(A)\rangle \times R), E_i)$ *is a strong subdigraph of* $\mathrm{Cay}(S, A)$ *with* $((g_i\langle p_1(A)\rangle \times R), E_i) \cong \mathrm{Cay}(\langle A\rangle, A)$ *for all* $i \in I$.

Lemma 2.2.8 ([44])**.** *Let $S = G \times R$ be a right group and A a nonempty subset of S. Then $\langle A\rangle = \langle p_1(A)\rangle \times p_2(A)$ is a right group contained in S.*

2.3 Domination Parameters

This section completely provides definitions of domination parameters consisting of the domination number and total domination number of digraphs.

Definition 2.3.1. Let $D = (V, E)$ be a digraph. A set $X \subseteq V$ of vertices in a digraph D is called a *dominating set* of D if for every vertex $v \in V \setminus X$, there exists $x \in X$ such that $(x, v) \in E$ and we say that x dominates v or v is dominated by x. The *domination number* $\gamma(D)$ of a digraph D is defined to be the minimum cardinality of a dominating set of D, that is,

$$\gamma(D) = \min\{|X| : X \text{ is a dominating set of } D\}.$$

A dominating set of D in which its cardinality equals $\gamma(D)$ is called a $\gamma-$*set* of D.

Definition 2.3.2. Let $D = (V, E)$ be a digraph. A set $X \subseteq V$ of vertices in a digraph D is called a *total dominating set* of D if for every vertex $v \in V$, there exists $x \in X$ such that $(x, v) \in E$. The *total domination number* $\gamma_t(D)$ of a digraph D is defined to be the minimum cardinality among all total dominating sets of D, that is,

$$\gamma_t(D) = \min\{|X| : X \text{ is a total dominating set of } D\}.$$

A total dominating set of D in which its cardinality equals $\gamma_t(D)$ is called a γ_t-*set* of D.

In general, every total dominating set of a digraph D is also a dominating set of D. This directly implies that $\gamma(D) \leq \gamma_t(D)$.

Example 2.3.1. Given a digraph D_2 of order 6 as shown in Figure 2.9.
We have $\gamma(D_2) = 2$ with the $\gamma-$set $\{1, 3\}$ and $\gamma_t(D_2) = 2$ with the γ_t-set $\{1, 4\}$.

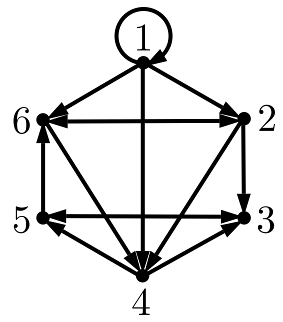

Figure 2.9: Digraph D_2.

Now, we consider the special group which is a subgroup of a symmetric group. For each $n \geq 3$, the *Dihedral group* D_n with an identity element e is a group of order $2n$ whose generators r and s satisfy:

$$r^n = e \ ; \ s^2 = e \ ; \ r^k \neq e \text{ if } 0 < k < n \ ; \ sr = r^{-1}s.$$

Therefore, we can take $D_n = \{e, r, r^2, r^3, \ldots, r^{n-1}, s, rs, r^2s, r^3s, \ldots, r^{n-1}s\}$.

The following lemmas give the results for the domination number and the total domination number of Cayley graphs of some groups with specific connection sets.

Lemma 2.3.2 ([9])**.** *Let $n \geq 3$ be an integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and k, t be integers such that $1 \leq k \leq m$ and $1 \leq t \leq n$. Let D_n be a Dihedral group and*

$$\Omega = \{r^{a_1}, r^{a_2}, \ldots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \ldots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \ldots, sr^{b_t}\} \subseteq D_n.$$

If $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$ and $d'_1 = b_1, d'_j = b_j - b_{j-1}$ for $2 \leq j \leq t$ and $d = \max\{d_i, d'_j\}$ where $1 \leq i \leq k$, $1 \leq j \leq t$, then $\gamma(\mathrm{Cay}(D_n, \Omega)) \leq 2d\lceil \frac{n}{2d+2a_k+b_t-b_1} \rceil$.

Lemma 2.3.3 ([9])**.** *Let $n \geq 3$ be an integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and k, t be integers such that $1 \leq k \leq m$ and $1 \leq t \leq n$ and d be a positive integer such that $d(2k + t + 1)$ divides n. Let D_n be a Dihedral group and*

$$\Omega = \{r^d, r^{2d}, \ldots, r^{kd}, r^{n-kd}, r^{n-(k-1)d}, \ldots, r^{n-d}, sr^d, sr^{2d}, \ldots, sr^{td}\} \subseteq D_n.$$

Then $\gamma(\mathrm{Cay}(D_n, \Omega)) = \frac{2n}{2k+t+1}$.

Lemma 2.3.4 ([9])**.** *Let $n \geq 3$ be an integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and k, t be integers such that $1 \leq k \leq m$ and $1 \leq t \leq n$. Let D_n be a Dihedral group and*

$$\Omega = \{r^{a_1}, r^{a_2}, \ldots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \ldots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \ldots, sr^{b_t}\} \subseteq D_n.$$

If $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$ and $d'_1 = b_1, d'_j = b_j - b_{j-1}$ for $2 \leq j \leq t$ and $d = \max\{d_i, d'_j\}$ where $1 \leq i \leq k$, $1 \leq j \leq t$, then $\gamma_t(\mathrm{Cay}(D_n, \Omega)) \leq 2d\lceil \frac{n}{d+2a_k} \rceil$.

Lemma 2.3.5 ([10]). *Let $n \geq 3$ be an odd integer, $m = \frac{n-1}{2}$ and*

$$A = \{m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\} \subseteq \mathbb{Z}_n$$

where $1 \leq k \leq m$. Then $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) = \lceil \frac{n}{2k} \rceil$.

Lemma 2.3.6 ([10]). *Let $n \geq 3$ be an even integer, $m = \lfloor \frac{n-1}{2} \rfloor$ and*

$$A = \{\tfrac{n}{2}, m, n-m, m-1, n-(m-1), \ldots, m-(k-1), n-(m-(k-1))\} \subseteq \mathbb{Z}_n$$

where $1 \leq k \leq m$. Then $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) = \lceil \frac{n}{2k+1} \rceil$.

2.4 Independence Parameters

The definitions about independence parameters on digraphs are presented in this section. Moreover, some new types of them concerning with dipaths are also prescribed.

Definition 2.4.1. For any digraph $D = (V, E)$, let u, v be two different vertices in V and I a nonempty subset of V. The vertex u is said to be *independent* to v (u, v are independent) if $(u, v) \notin E$ and $(v, u) \notin E$. The set I is called an *independent set* of D if any two different vertices in I are independent. The *independence number* of D is the maximum cardinality of an independent set of D and denoted by $\alpha(D)$, that is,

$$\alpha(D) = \max\{|I| : I \text{ is an independent set of } D\}.$$

An independent set of D in which its cardinality equals $\alpha(D)$ is called an $\alpha-$*set* of D.

Definition 2.4.2. For any digraph $D = (V, E)$, let u, v be two different vertices in V and I a nonempty subset of V. The vertex u is said to be *weakly independent* to v (u, v are weakly independent) if $(u, v) \notin E$ or $(v, u) \notin E$. The set I is called a *weakly independent set* of D if any two different vertices in I are weakly independent. The *weakly independence number* of D is the maximum cardinality of a weakly independent set of D and denoted by $\alpha_w(D)$, that is,

$$\alpha_w(D) = \max\{|I| : I \text{ is a weakly independent set of } D\}.$$

A weakly independent set of D in which its cardinality equals $\alpha_w(D)$ is called an α_w-*set* of D.

Definition 2.4.3. For any digraph $D = (V, E)$, let u, v be two different vertices in V and I a nonempty subset of V. The vertex u is said to be *dipath independent* to v (u, v are dipath independent) if there is no dipath from u to v and there is no dipath from v to u. The set I is called a *dipath independent set* of D if any two different vertices in I are dipath independent. The *dipath independence number* of D is the maximum cardinality of a dipath independent set of D and denoted by $\alpha_p(D)$, that is,

$$\alpha_p(D) = \max\{|I| : I \text{ is a dipath independent set of } D\}.$$

A dipath independent set of D in which its cardinality equals $\alpha_p(D)$ is called an α_p*−set* of D.

Definition 2.4.4. For any digraph $D = (V, E)$, let u, v be two different vertices in V and I a nonempty subset of V. The vertex u is said to be *weakly dipath independent* to v (u, v are weakly dipath independent) if there is no dipath from u to v or there is no dipath from v to u. The set I is called a *weakly dipath independent set* of D if any two different vertices in I are weakly dipath independent. The *weakly dipath independence number* of D is the maximum cardinality of a weakly dipath independent set of D and denoted by $\alpha_{wp}(D)$, that is,

$$\alpha_{wp}(D) = \max\{|I| : I \text{ is a weakly dipath independent set of } D\}.$$

A weakly dipath independent set of D in which its cardinality equals $\alpha_{wp}(D)$ is called an α_{wp}*−set* of D.

In fact, every (dipath) independent set of a digraph D is also a weakly (dipath) independent set of D, so we can conclude that $\alpha(D) \leq \alpha_w(D)$ $(\alpha_p(D) \leq \alpha_{wp}(D))$.

Example 2.4.1. Consider the Digraph D_2 (shown in Figure 2.9) which is shown as follows.

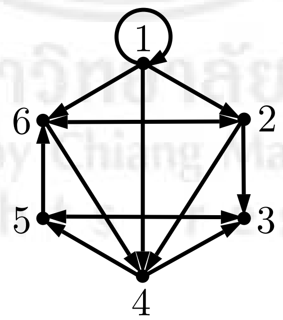

Figure 2.10: Digraph D_2.

We obtain that $\alpha(D_2) = 2$ with the α−set $\{2, 5\}$; $\alpha_w(D_2) = 4$ with the α_w−set $\{1, 2, 3, 4\}$; $\alpha_p(D_2) = 1$ with the α_p−set $\{1\}$; and $\alpha_{wp}(D_2) = 2$ with the α_{wp}−set $\{1, 2\}$.

2.5 Independent Domination Parameters

Now, we introduce some special types of domination parameters which are known as independent domination parameters of digraphs.

Definition 2.5.1. Let $D = (V, E)$ be a digraph and I a nonempty subset of V. The set I is called an *independent dominating set* of D if I is a dominating set of D and independent in D. The *independent domination number* of D, denoted by $i(D)$, is the minimum cardinality of an independent dominating set of D, that is,

$$i(D) = \min\{|I| : I \text{ is an independent dominating set of } D\}.$$

An independent dominating set of D in which its cardinality equals $i(D)$ is called an $i-set$ of D.

Definition 2.5.2. Let $D = (V, E)$ be a digraph and I a nonempty subset of V. The set I is called a *weakly independent dominating set* of D if I is a dominating set of D and weakly independent in D. The *weakly independent domination number* of D, denoted by $i_w(D)$, is the minimum cardinality of a weakly independent dominating set of D, that is,

$$i_w(D) = \min\{|I| : I \text{ is a weakly independent dominating set of } D\}.$$

A weakly independent dominating set of D in which its cardinality equals $i_w(D)$ is called an i_w-set of D.

Example 2.5.1. Let D_3 be a digraph of order 6 given as follows.

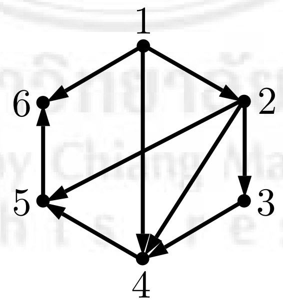

Figure 2.11: Digraph D_3.

We obtain that $i(D_3) = 3$ with the $i-$set $\{1, 3, 5\}$ and $i_w(D_3) = 2$ with the i_w-set $\{1, 2\}$.

Definition 2.5.3. Let $D = (V, E)$ be a digraph and I a nonempty subset of V. The set I is called a *dipath independent dominating set* of D if I is a dominating set of D and dipath independent in D. The *dipath independent domination number* of D, denoted by $i_p(D)$, is the minimum cardinality of a dipath independent dominating set of D, that is,

$$i_p(D) = \min\{|I| : I \text{ is a dipath independent dominating set of } D\}.$$

A dipath independent dominating set of D in which its cardinality equals $i_p(D)$ is called an i_p*−set* of D.

Definition 2.5.4. Let $D = (V, E)$ be a digraph and I a nonempty subset of V. The set I is called a *weakly dipath independent dominating set* of D if I is a dominating set of D and weakly dipath independent in D. The *weakly dipath independent domination number* of D, denoted by $i_{wp}(D)$, is the minimum cardinality of a weakly dipath independent dominating set of D, that is,

$$i_{wp}(D) = \min\{|I| : I \text{ is a weakly dipath independent dominating set of } D\}.$$

A weakly dipath independent dominating set of D in which its cardinality equals $i_{wp}(D)$ is called an i_{wp}*−set* of D.

Example 2.5.2. Consider the digraph D_4 of order 6 illustrated as follows.

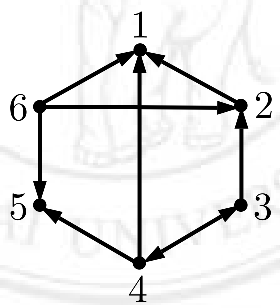

Figure 2.12: Digraph D_4.

We obtain that $i_p(D_4) = 2$ with the i_p−set $\{4, 6\}$ and $i_{wp}(D_4) = 2$ with the i_{wp}−set $\{3, 6\}$.