CHAPTER 3

Basic Properties of Cayley Digraphs of Rectangular Groups

We now give some characterizations of Cayley digraphs of rectangular groups with respect to their connection sets. Moreover, some basic properties of Cayley digraphs Δ of rectangular groups $G \times L \times R$ with connection sets A are also presented.

3.1 Characterizations of Cayley Digraphs of Rectangular Groups

Theorem 3.1.1. *If* $\overline{A} = \{(a, \alpha) \in G \times R : (a, l, \alpha) \in A \text{ for some } l \in L\}$*, then* Δ *is the disjoint union of* $|L|$ *strong subdigraphs which each subdigraph is isomorphic to the Cayley digraph* $\mathrm{Cay}(G \times R, \overline{A})$ *of a right group* $G \times R$ *with a connection set* $\overline{A}$*.*

Proof. Let $\overline{A} = \{(a, \alpha) \in G \times R : (a, l, \alpha) \in A \text{ for some } l \in L\}$ be a connection set of the Cayley digraph $\mathrm{Cay}(G \times R, \overline{A})$ of a right group $G \times R$. For each $l \in L$, let $(G \times \{l\} \times R, E_l)$ denote a strong subdigraph of Δ induced by $G \times \{l\} \times R$ as a vertex set and its arc set is E_l. We first show that $\Delta = \dot{\bigcup\limits_{l \in L}} (G \times \{l\} \times R, E_l)$. It is easy to obtain that $\dot{\bigcup\limits_{l \in L}} (G \times \{l\} \times R, E_l)$ is a subdigraph of Δ. Now, let $(g, k, \alpha) \in V(\Delta)$. It follows that

$$(g, k, \alpha) \in G \times \{k\} \times R = V(G \times \{k\} \times R, E_k) \subseteq V\left(\dot{\bigcup_{l \in L}} (G \times \{l\} \times R, E_l)\right), \text{ obviously.}$$

Next, let $((g_1, l_1, \alpha_1), (g_2, l_2, \alpha_2)) \in E(\Delta)$. Hence there exists $(a, m, \lambda) \in A$ such that

$$(g_2, l_2, \alpha_2) = (g_1, l_1, \alpha_1)(a, m, \lambda) = (g_1 a, l_1, \lambda), \text{ that is, } l_1 = l_2.$$

Thus $((g_1, l_1, \alpha_1), (g_2, l_2, \alpha_2)) \in E(G \times \{l_1\} \times R, E_{l_1}) \subseteq E\left(\dot{\bigcup\limits_{l \in L}} (G \times \{l\} \times R, E_l)\right)$. It is not difficult to verify that $(G \times \{t_1\} \times R, E_{t_1})$ is disjoint from $(G \times \{t_2\} \times R, E_{t_2})$ for all different elements t_1 and t_2 of L. So we can conclude that $\Delta = \dot{\bigcup\limits_{l \in L}} (G \times \{l\} \times R, E_l)$.

Let $l \in L$ be fixed. We next show that $(G \times \{l\} \times R, E_l) \cong \mathrm{Cay}(G \times R, \overline{A})$. Let $\Phi : G \times \{l\} \times R \to G \times R$ be a mapping defined by

$$\Phi(g, l, \alpha) = (g, \alpha) \text{ for all } (g, l, \alpha) \in G \times \{l\} \times R.$$

Then Φ is a well-defined bijection followed from the straightforward proof. We only need to show that Φ and Φ^{-1} are arc-preserving. Let $((g_1, l, \alpha_1), (g_2, l, \alpha_2)) \in E(G \times \{l\} \times R, E_l)$.

Hence $(g_2, l, \alpha_2) = (g_1, l, \alpha_1)(a, k, \alpha) = (g_1 a, l, \alpha)$ for some $(a, k, \alpha) \in A$. Thus $(a, \alpha) \in \overline{A}$ and $\Phi(g_2, l, \alpha_2) = (g_2, \alpha_2) = (g_1 a, \alpha) = (g_1, \alpha_1)(a, \alpha) = \Phi(g_1, l, \alpha_1)(a, \alpha)$ which implies that $(\Phi(g_1, l, \alpha_1), \Phi(g_2, l, \alpha_2)) \in E(\mathrm{Cay}(G \times R, \overline{A}))$. We next consider $((h_1, \beta_1), (h_2, \beta_2)) \in E(\mathrm{Cay}(G \times R, \overline{A}))$. Then there exists $(b, \beta) \in \overline{A}$ such that $(h_2, \beta_2) = (h_1, \beta_1)(b, \beta) = (h_1 b, \beta)$. We obtain that $(b, j, \beta) \in A$ for some $j \in L$ and

$$\Phi^{-1}(h_2, \beta_2) = (h_2, l, \beta_2) = (h_1 b, l, \beta) = (h_1, l, \beta_1)(b, j, \beta) = \Phi^{-1}(h_1, \beta_1)(b, j, \beta)$$

and hence $(\Phi^{-1}(h_1, \beta_1), \Phi^{-1}(h_2, \beta_2)) \in E(G \times \{l\} \times R, E_l)$. Therefore, both Φ and Φ^{-1} are arc-preserving. Consequently, Φ is an isomorphism that means $(G \times \{l\} \times R, E_l) \cong \mathrm{Cay}(G \times R, \overline{A})$, this completes the proof of our assertion. □

By applying Theorem 3.1.1 and Lemma 2.2.7 under the condition $p_3(A) = R$, we can simply obtain the following corollary.

Corollary 3.1.2. *If $p_3(A) = R$, then Δ is the disjoint union of $\frac{|G||L|}{|\langle p_1(A)\rangle|}$ strong subdigraphs which each subdigraph is isomorphic to $\mathrm{Cay}(\langle \overline{A}\rangle, \overline{A})$ where $\overline{A}$ is the connection set given in the above theorem.*

3.2 Some Properties of Cayley Digraphs of Rectangular Groups

Let (V, E) be a digraph and u, v be any two vertices in V. Recall that the digraph (V, E) is said to be *strongly connected* if there is a dipath in (V, E) connecting from u to v. The following theorem gives the sufficient condition for $\mathrm{Cay}(\langle \overline{A}\rangle, \overline{A})$ to be strongly connected where $\overline{A} = \{(a, \alpha) \in G \times R : (a, l, \alpha) \in A \text{ for some } l \in L\}$.

Theorem 3.2.1. *If $p_3(A) = R$, then $\mathrm{Cay}(\langle \overline{A}\rangle, \overline{A})$ is strongly connected.*

Proof. Let A be a connection set of Δ such that $p_3(A) = R$. In order to show that $\mathrm{Cay}(\langle \overline{A}\rangle, \overline{A})$ is strongly connected, we let (x, λ) and (y, μ) be any two vertices of $\mathrm{Cay}(\langle \overline{A}\rangle, \overline{A})$, that is, $(x, \lambda), (y, \mu) \in \langle \overline{A}\rangle$. From $p_3(A) = R$, we also get that $p_2(\overline{A}) = R$. By Lemma 2.2.8, we have $\langle \overline{A}\rangle = \langle p_1(\overline{A})\rangle \times R$. Since $\mu \in R$, there exists $a \in p_1(\overline{A})$ such that $(a, \mu) \in \overline{A}$. Since $x, y \in p_1(\langle \overline{A}\rangle) = \langle p_1(\overline{A})\rangle$ and $\langle p_1(\overline{A})\rangle$ is a group, we can write $y = xu$ for some $u \in \langle p_1(\overline{A})\rangle$. Hence $u = u_1 u_2 \cdots u_t$ where $u_1, u_2, \ldots, u_t \in p_1(\overline{A})$. Thus there exist $\omega_1, \omega_2, \ldots, \omega_t \in R$ in which $(u_1, \omega_1), (u_2, \omega_2), \ldots, (u_t, \omega_t) \in \overline{A}$. Consider

$$(y, \omega_t) = (xu, \omega_t) = (xu_1 u_2 \cdots u_t, \omega_t) = (x, \lambda)(u_1, \omega_1)(u_2, \omega_2) \cdots (u_t, \omega_t),$$

we conclude that there exists a dipath from (x, λ) through to (y, ω_t). Assume that $|a| = n$, the order of an element a in a group $\langle p_1(\overline{A})\rangle$, for some $n \in \mathbb{N}$. Therefore, $y = ya^n$ and then $(y, \mu) = (ya^n, \mu) = (y, \omega_t)(a^n, \mu) = (y, \omega_t)(a, \mu)^n$ which implies that there exists a dipath connecting from (y, ω_t) to (y, μ). So we can conclude that there exists a dipath from (x, λ) to (y, μ) which leads to the result that $\mathrm{Cay}(\langle\overline{A}\rangle, \overline{A})$ is strongly connected, as required. $\square$

Proposition 3.2.2. *If A is any connection set of Δ, then $\langle p_1(A)\rangle = p_1(\langle A\rangle)$.*

Proof. Let A be a connection set of Δ and $a \in \langle p_1(A)\rangle$. Then $a = a_1a_2\cdots a_k$ for some $a_1, a_2, \ldots, a_k \in p_1(A)$. So there exist $l_1, l_2, \ldots, l_k \in L$ and $\gamma_1, \gamma_2, \ldots, \gamma_k \in R$ such that $(a_1, l_1, \gamma_1), (a_2, l_2, \gamma_2), \ldots, (a_k, l_k, \gamma_k) \in A$. Consider

$$(a, l_1, \gamma_k) = (a_1a_2\cdots a_k, l_1, \gamma_k) = (a_1, l_1, \gamma_1)(a_2, l_2, \gamma_2)\cdots(a_k, l_k, \gamma_k) \in \langle A\rangle,$$

we can conclude that $a \in p_1(\langle A\rangle)$. Thus $\langle p_1(A)\rangle \subseteq p_1(\langle A\rangle)$. In order to prove the reverse inclusion, let $b \in p_1(\langle A\rangle)$. Then there exist $m \in L$ and $\mu \in R$ in which $(b, m, \mu) \in \langle A\rangle$. We can obtain that

$$(b, m, \mu) = (b_1, m_1, \mu_1)(b_2, m_2, \mu_2)\cdots(b_n, m_n, \mu_n) = (b_1b_2\cdots b_n, m_1, \mu_n)$$

where $(b_1, m_1, \mu_1), (b_2, m_2, \mu_2), \ldots, (b_n, m_n, \mu_n) \in A$. Hence $b = b_1b_2\cdots b_n \in \langle p_1(A)\rangle$ in which $b_1, b_2, \ldots, b_n \in p_1(A)$. Therefore, $p_1(\langle A\rangle) \subseteq \langle p_1(A)\rangle$ and whence $\langle p_1(A)\rangle = p_1(\langle A\rangle)$, as desired. $\square$

The above results are valuable for proving other results of this thesis which will be presented in the sequel.