CHAPTER 4

Domination on Cayley Digraphs of Rectangular Groups

In this chapter, we provide the results of domination parameters consisting of the domination number and total domination number of Cayley digraphs of rectangular groups with their connection sets. In particular, since left groups and right groups are rectangular groups, we also consider such semigroups here. Furthermore, we present the value or bounds for the domination number of Cayley digraphs of left groups and right groups. Some examples which give sharpness of those bounds are also shown. Moreover, we consider the total domination number and give the necessary and sufficient conditions for the existence of total dominating sets in Cayley digraphs of left groups and right groups.

4.1 Domination Number

We start with the result of the domination number on Cayley digraphs of rectangular groups in the term of the domination number of Cayley digraphs of right groups with corresponding connection sets.

Theorem 4.1.1. *Let A be a connection set of Δ. If $\overline{A} = \{(g,r) \in G \times R : (g,l,r) \in A$ for some $l \in L\}$, then $\gamma(\Delta) = |L| \cdot \gamma(\mathrm{Cay}(G \times R, \overline{A}))$.*

Proof. From the characterization of Cayley digraphs of rectangular groups described in Theorem 3.1.1, we get that Δ is the disjoint union of $|L|$ isomorphic subdigraphs which each subdigraph is isomorphic to the Cayley digraph $\mathrm{Cay}(G \times R, \overline{A})$ of a right group $G \times R$ with the connection set $\overline{A} = \{(a, \alpha) \in G \times R : (a, l, \alpha) \in A$ for some $l \in L\}$. Let $|L| = m$ for some $m \in \mathbb{N}$ and $I = \{1, 2, \dots, m\}$. Suppose that those m subdigraphs of Δ are $D_1, D_2, \dots, D_m$. Consequently,

$$\gamma(\Delta) = \gamma\left(\dot{\bigcup_{i \in I}} D_i\right) = \gamma\left(\dot{\bigcup_{i \in I}} \mathrm{Cay}(G \times R, \overline{A})\right) = \sum_{i=1}^{m} \gamma(\mathrm{Cay}(G \times R, \overline{A}))$$
$$= m \cdot \gamma(\mathrm{Cay}(G \times R, \overline{A})) = |L| \cdot \gamma(\mathrm{Cay}(G \times R, \overline{A})). \qquad \square$$

Next, we will consider the Cayley digraph Γ of a left group $G \times L$ with a connection set A. The following result gives the domination number of a Cayley digraph of a left group in the term of the domination number of a Cayley digraph of its subgroup.

Theorem 4.1.2. *Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$ for some $k \in \mathbb{N}$. Then $\gamma(\Gamma) = k \cdot |L| \cdot \gamma(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$ where A is a connection set of Γ.*

Proof. Let $I = \{1, 2, \ldots, k\}$. By Lemma 2.2.6, we have $\Gamma = \dot{\bigcup}_{i\in I, l\in L} ((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ where $((g_i\langle p_1(A)\rangle \times \{l\}), E_{il})$ is a strong subdigraph of Γ such that

$$((g_i\langle p_1(A)\rangle \times \{l\}), E_{il}) \cong \mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)) \text{ for all } i \in I, l \in L.$$

Therefore, $\gamma(\Gamma) = \sum_{i\in I}\sum_{l\in L} \gamma((g_i\langle p_1(A)\rangle \times \{l\}), E_{il}) = |I| \cdot |L| \cdot \gamma(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$. □

Sometimes, it is not easy to find the value of $\gamma(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$, so we can not find $\gamma(\Gamma)$ actually. However, we can know the bounds of $\gamma(\Gamma)$ which does not depend on $\gamma(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$. The next theorem gives the bounds of the domination number in Cayley digraphs of left groups with their corresponding connection sets.

Theorem 4.1.3. *Let $G \times L$ be a left group and A a connection set of Γ such that the identity of G lies in $p_1(A)$. If H is a subgroup of G with a maximum cardinality that contained in $p_1(A)$, then $\frac{|G|}{|p_1(A)|}|L| \leq \gamma(\Gamma) \leq [G : H]|L|$ where $[G : H]$ is the index of H in G.*

Proof. Suppose that H is the subgroup of G with a maximum cardinality such that $H \subseteq p_1(A)$. We will show that $\gamma(\Gamma) \leq [G : H]|L|$. Let $[G : H] = k$ for some $k \in \mathbb{N}$. Consider the set of all left cosets of H in G, $\{g_1H, g_2H, \ldots, g_kH\}$. Pick one element from each left coset $g_1H, g_2H, \ldots, g_kH$, say that $g_1h_1, g_2h_2, \ldots, g_kh_k$, respectively. Let $l \in L$ and D denote $\mathrm{Cay}(G\times\{l\}, p_1(A)\times\{l\})$ and $Y = \{g_1h_1, g_2h_2, \ldots, g_kh_k\} \times \{l\} \subseteq G \times \{l\}$. We will prove that Y is a dominating set of D. Let $(g, l) \in (G \times \{l\}) \setminus Y$. Since $g \in G = \bigcup_{t=1}^{k} g_tH$, we get that $g \in g_jH$ for some $1 \leq j \leq k$. Then $g = g_jh$ for some $h \in H$. Thus $(g_jh_j, l) \in Y$ and $h_j^{-1}h \in H \subseteq p_1(A)$. So there exists $q \in p_2(A)$ such that $(h_j^{-1}h, q) \in A$ and we have $(g, l) = (g_jh, l) = ((g_jh_j)(h_j^{-1}h), l) = (g_jh_j, l)(h_j^{-1}h, q) \in YA$. Hence Y is the dominating set of D and then $\gamma(D) \leq |Y| = k = [G : H]$. By Lemma 2.2.5, we can conclude that $\gamma(\Gamma) = \gamma(D)|L| \leq [G : H]|L|$. Now, we will prove that $\gamma(\Gamma) \geq \frac{|G|}{|p_1(A)|}|L|$. By Lemma 2.2.5(1), we obtain that

$$\mathrm{Cay}(G \times \{l_1\}, p_1(A) \times \{l_1\}) \cong \mathrm{Cay}(G \times \{l_2\}, p_1(A) \times \{l_2\}) \text{ for all } l_1, l_2 \in L.$$

For each $l \in L$, we will consider the domination number of D, the digraph defined as above, and let X be the dominating set of D such that X is a $\gamma-$set. Since the identity of

G lies in $p_1(A)$ and X is the dominating set of D, we get that $(X)(p_1(A) \times \{l\}) = G \times \{l\}$. Hence $|G| = |G \times \{l\}| = |(X)(p_1(A) \times \{l\})| \leq |X||p_1(A) \times \{l\}| = |X||p_1(A)|$. Therefore, $\gamma(D) = |X| \geq \frac{|G|}{|p_1(A)|}$. By Lemma 2.2.5(2), we have $\gamma(\Gamma) = \gamma(D)|L| \geq \frac{|G|}{|p_1(A)|}|L|$. □

Corollary 4.1.4. *Let $G \times L$ be a left group and $A = K \times L$ a connection set of Γ where K is any subgroup of G. Then $\gamma(\Gamma) = [G : K]|L|$.*

Proof. Since $A = K \times L$ where K is any subgroup of G, we obtain that the identity e of G lies in $K = p_1(A)$. Moreover, we get that K is the subgroup of G with a maximum cardinality that contained in $p_1(A)$. By applying Theorem 4.1.3, we can conclude that $\frac{|G|}{|K|}|L| \leq \gamma(\Gamma) \leq [G : K]|L|$. Therefore, $\gamma(\Gamma) = [G : K]|L|$ since $[G : K] = \frac{|G|}{|K|}$. □

The following example gives the sharpness of the bounds given in Theorem 4.1.3.

Example 4.1.5. Let $\mathbb{Z}_6 \times L$ be a left group where $\mathbb{Z}_6$ is a group of integers modulo 6 under the addition and $L = \{l_1, l_2\}$ is a left zero semigroup.

(1). Consider the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_6 \times L, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.

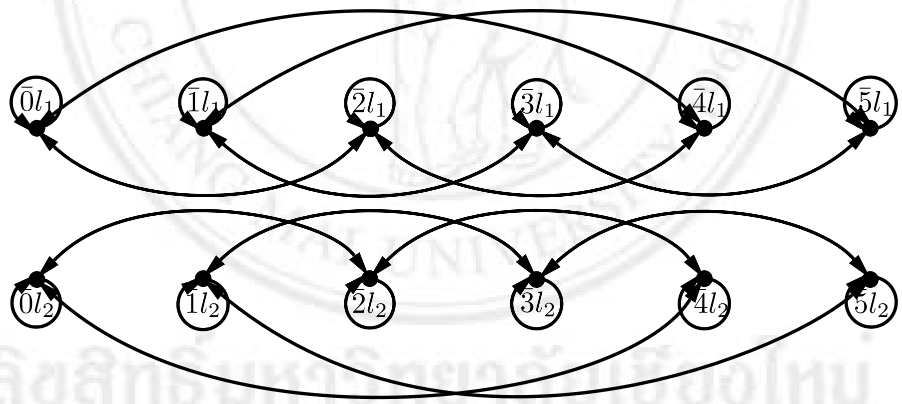

Figure 4.1: $\mathrm{Cay}(\mathbb{Z}_6 \times L, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.

We obtain that $X = \{(\bar{0}, l_1), (\bar{0}, l_2), (\bar{1}, l_1), (\bar{1}, l_2)\}$ is a $\gamma-$set of the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.
Thus $\gamma(\mathrm{Cay}(\mathbb{Z}_6 \times L_2, \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})) = |X| = 4 = 2(2) = [\mathbb{Z}_6 : H]|L_2|$ where $H = \{\bar{0}, \bar{2}, \bar{4}\}$ is the subgroup with a maximum cardinality of $\mathbb{Z}_6$ that contained in $p_1(\{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1)\})$.
Similarly, if $A = \{(\bar{0}, l_1), (\bar{2}, l_1), (\bar{4}, l_1), \ldots, (\overline{2k-2}, l_1)\}$ is a nonempty subset of $\mathbb{Z}_{2k} \times L$ where $k \in \mathbb{N}$ and $L = \{l_1, l_2\}$, then the set $\{(\bar{0}, l_1), (\bar{0}, l_2), (\bar{1}, l_1), (\bar{1}, l_2)\}$ is a $\gamma-$set of a

Cayley digraph $\mathrm{Cay}(\mathbb{Z}_{2k} \times L, A)$. Hence $\gamma(\mathrm{Cay}(\mathbb{Z}_{2k} \times L, A)) = 4 = [\mathbb{Z}_{2k} : H]|L|$ where $H = \{\overline{0}, \overline{2}, \overline{4}, \ldots, \overline{2k-2}\}$ is the subgroup with the maximum cardinality of $\mathbb{Z}_{2k}$ which is contained in $p_1(A)$.

(2). Consider the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_6 \times L, \{(\overline{0}, l_1), (\overline{3}, l_2)\})$.

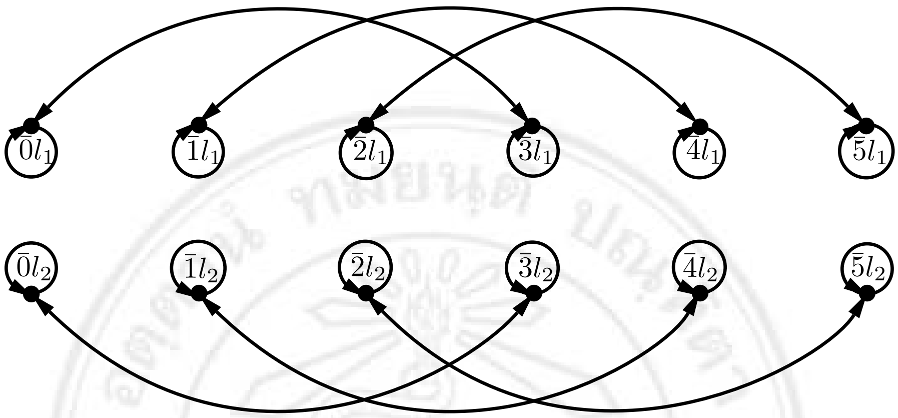

Figure 4.2: $\mathrm{Cay}(\mathbb{Z}_6 \times L, \{(\overline{0}, l_1), (\overline{3}, l_2)\})$.

We obtain that $Y = \{(\overline{0}, l_1), (\overline{1}, l_1), (\overline{2}, l_1), (\overline{0}, l_2), (\overline{1}, l_2), (\overline{2}, l_2)\}$ is a $\gamma-$set of the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_6 \times L, \{(\overline{0}, l_1), (\overline{3}, l_2)\})$ and $\gamma(\mathrm{Cay}(\mathbb{Z}_6 \times L, \{(\overline{0}, l_1), (\overline{3}, l_2)\})) = |Y| = 6 = \frac{6}{2} \times 2 = \frac{|\mathbb{Z}_6|}{|p_1(\{(\overline{0}, l_1), (\overline{3}, l_2)\})|} \times |L|$.
Similarly, if $A = \{(\overline{0}, l_1), (\overline{k}, l_2)\}$ is a nonempty subset of $\mathbb{Z}_{2k} \times L$ where $k \in \mathbb{N}$ and $L = \{l_1, l_2\}$, then the set $\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{k-1}\} \times \{l_1, l_2\}$ is a $\gamma-$set of $\mathrm{Cay}(\mathbb{Z}_{2k} \times L, A)$. Hence $\gamma(\mathrm{Cay}(\mathbb{Z}_{2k} \times L, A)) = 2k = \frac{2k}{2} \times 2 = \frac{|\mathbb{Z}_{2k}|}{|p_1(A)|} \times |L|$.

The following two propositions present the results for the domination numbers of Cayley digraphs of left groups which are isomorphic to the direct product of the $2n-$element Dihedral group D_n and a left zero semigroup L with suitable connection sets.

Proposition 4.1.6. *Let $S = D_n \times L$ be a left group and A a nonempty subset of S such that the identity element $e \notin p_1(A)$ and for each $x \in p_1(A)$, x^{-1} must belong to $p_1(A)$. Let $n \geq 3$ be an integer, $c = \lfloor \frac{n-1}{2} \rfloor$ and k, t be integers such that $1 \leq k \leq c$ and $1 \leq t \leq n$. Let $p_1(A) = \{r^{a_1}, r^{a_2}, \ldots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \ldots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \ldots, sr^{b_t}\} \subseteq D_n$.
If $d = \max\{a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, b_1, b_2 - b_1, b_3 - b_2, \ldots, b_t - b_{t-1}\}$, then $\gamma(\mathrm{Cay}(S, A)) \leq 2d|L|\lceil \frac{n}{2d+2a_k+b_t-b_1} \rceil$.*

Proof. This proposition follows from Lemma 2.2.5 and Lemma 2.3.2. □

Proposition 4.1.7. *Let $S = D_n \times L$ be a left group and A a nonempty subset of S such that the identity element $e \notin p_1(A)$ and for each $x \in p_1(A)$, x^{-1} must belong to $p_1(A)$. Let $n \geq 3$ be an integer, $c = \lfloor \frac{n-1}{2} \rfloor$ and k, t be integers such that $1 \leq k \leq c$ and $1 \leq t \leq n$. Let $p_1(A) = \{r^d, r^{2d}, \ldots, r^{kd}, r^{n-kd}, r^{n-(k-1)d}, \ldots, r^{n-d}, sr^d, sr^{2d}, \ldots, sr^{td}\} \subseteq D_n$. If d is an integer such that $d(2k+t+1)$ divides n, then $\gamma(\text{Cay}(S, A)) = \frac{2n|L|}{2k+t+1}$.*

Proof. This proposition is a direct result from Lemma 2.2.5 and Lemma 2.3.3. □

Now, we show other results of the domination number of Cayley digraphs of $\mathbb{Z}_n$, the group of integers modulo n, with a connection set $\{\overline{1}, \overline{t}\} \subseteq \mathbb{Z}_n$ for applying the results to the domination number of Cayley digraphs of a left group $\mathbb{Z}_n \times L$ where L is a left zero semigroup. Furthermore, let (V, E) be a digraph and for each $x \in V$, we define $N(x) = \{y \in V : (x, y) \in E\}$ to be the set of all neighbours of x and let $N[x] = N(x) \cup \{x\}$.

In general, it is easy to verify that $\lceil \frac{n}{3} \rceil \leq \gamma(\text{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{t}\})) \leq \lceil \frac{n}{2} \rceil$.

Proposition 4.1.8. *Let $n \geq 2$ be a positive integer. Then $\gamma(\text{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{2}\})) = \lceil \frac{n}{3} \rceil$.*

Proof. We will consider the case $n \equiv 1 \pmod 3$.
It is easy to see that $\{\overline{1}, \overline{4}, \overline{7}, \ldots, \overline{n-3}, \overline{n}\}$ is a dominating set of $\text{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{2}\})$.
Hence $\gamma(\text{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{2}\})) \leq |\{\overline{1}, \overline{4}, \overline{7}, \ldots, \overline{n-3}, \overline{n}\}| = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil$.
Suppose that there exists a dominating set X such that $|X| < \frac{n+2}{3}$, that is, $|X| \leq \frac{n-1}{3}$. Since $|N[x]| \leq 3$ for all $x \in X$, we obtain that $|\bigcup_{x \in X} N[x]| \leq 3|X| \leq n - 1 < n$ which is a contradiction. Therefore, $\gamma(\text{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{2}\})) = \lceil \frac{n}{3} \rceil$. Other cases can be proved by using the similar arguments of the above case. □

Lemma 4.1.9. *Let $n \geq 3$ be a positive integer and X a dominating set of $\text{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})$. For each $x \in X, |N[x] \cap N[v]| \geq 1$ for some $v \in X \setminus \{x\}$.*

Proof. Let X be a dominating set of $\text{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})$ and $x \in X$.
Then $N[x] = \{x, x+1, x+3\}$. Since $x + 2 \notin N[x]$ and $x + 2$ has to be dominated, we can conclude that $x + 2 \in X$ or $x + 2 \in N[y]$ for some $y \in X$.

If $x + 2 \in X$, then $N[x+2] = \{x+2, x+3, x+5\}$, that is, $x + 3 \in N[x] \cap N[x+2]$ which implies that $|N[x] \cap N[x+2]| \geq 1$.

If $x + 2 \in N[y]$, then $y = x + 1$ or $y = x - 1$.

If $y = x + 1$, then $x + 1 \in X$. Thus $x + 1 \in N[x] \cap N[x+1]$ which leads to the fact that $|N[x] \cap N[x+1]| \geq 1$.

If $y = x - 1$, then $x - 1 \in X$. Thus $x \in N[x] \cap N[x-1]$ which implies that $|N[x] \cap N[x-1]| \geq 1$. $\square$

Proposition 4.1.10. *Let $n \geq 3$ be a positive integer.*

Then $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})) = \begin{cases} 2\lceil \frac{n}{5} \rceil - 1 & \text{if } n \equiv 1, 2 \pmod 5, \\ 2\lceil \frac{n}{5} \rceil & \text{if } n \equiv 0, 3, 4 \pmod 5. \end{cases}$

Proof. We will consider the case $n \equiv 1 \pmod 5$. In this case, we can conclude that $T = \{\overline{1}, \overline{2}, \overline{6}, \overline{7}, \overline{11}, \overline{12}, \ldots, \overline{n-5}, \overline{n-4}, \overline{n}\}$ is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})$ which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})) \leq |T| = \frac{2n+3}{5} = 2\lceil \frac{n}{5} \rceil - 1$. Next, suppose that there exists a dominating set X such that $|X| \leq 2\lceil \frac{n}{5} \rceil - 2 = \frac{2(n-1)}{5}$. For each $x \in X$, we have by Lemma 4.1.9 that $N[x] \cap N[y] \geq 1$ for some $y \in X \setminus \{x\}$. Since $|N[x]| \leq 3$, we have $|\bigcup_{x \in X} N[x]| \leq 3|X| - \lceil \frac{|X|}{2} \rceil \leq \frac{5|X|}{2} \leq n - 1 < n$, that is, $\bigcup_{x \in X} N[x] \subsetneq \mathbb{Z}_n$. Hence X does not dominate $\mathbb{Z}_n$ which is a contradiction. Therefore, $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})) = |T| = 2\lceil \frac{n}{5} \rceil - 1$. Similarly, we can prove the case $n \equiv 2 \pmod 5$.

Now, we will consider the case $n \equiv 3 \pmod 5$. For this case, we can obtain that $T = \{\overline{1}, \overline{2}, \overline{6}, \overline{7}, \overline{11}, \overline{12}, \ldots, \overline{n-2}, \overline{n-1}\}$ is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})$. Then $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})) \leq |T| = \frac{2n+4}{5} = 2\lceil \frac{n}{5} \rceil$. Assume to the contrary that there exists a dominating set X such that $|X| \leq 2\lceil \frac{n}{5} \rceil - 1 = \frac{2n-1}{5}$. Again by Lemma 4.1.9, we have $|\bigcup_{x \in X} N[x]| \leq \frac{5|X|}{2} \leq \frac{2n-1}{2} < \frac{2n}{2} = n$. Whence, X does not dominate $\mathbb{Z}_n$ which contradicts to the property of the dominating set X. So we can conclude that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{3}\})) = |T| = 2\lceil \frac{n}{5} \rceil$. For the cases $n \equiv 0, 4 \pmod 5$, we can prove them, similarly. $\square$

Proposition 4.1.11. *Let $n \geq 4$ be a positive integer.*

Then $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq \begin{cases} 3\lceil \frac{n}{7} \rceil & \text{if } n \equiv 0, 6 \pmod 7, \\ 3\lceil \frac{n}{7} \rceil - 1 & \text{if } n \equiv 4, 5 \pmod 7, \\ 3\lceil \frac{n}{7} \rceil - 2 & \text{if } n \equiv 1, 2, 3 \pmod 7. \end{cases}$

Proof. Let $n \geq 4$ be a positive integer.

For $n \equiv 0 \pmod 7$, we obtain that X_0 is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})$ where
$X_0 = \{\overline{1}, \overline{8}, \overline{15}, \overline{22}, \ldots, \overline{n-6}\} \cup \{\overline{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-4}\} \cup \{\overline{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-1}\}$
which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq |X_0| = 3\lceil \frac{n}{7} \rceil$.

For $n \equiv 1 \pmod 7$, we obtain that X_1 is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})$ where
$X_1 = \{\overline{1}, \overline{8}, \overline{15}, \overline{22}, \ldots, \overline{n}\} \cup \{\overline{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-5}\} \cup \{\overline{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-2}\}$
which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq |X_1| = 3\lceil \frac{n}{7} \rceil - 2$.

For $n \equiv 2 \pmod 7$, we obtain that X_2 is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})$ where
$X_2 = \{\overline{1}, \overline{8}, \overline{15}, \overline{22}, \ldots, \overline{n-1}\} \cup \{\overline{3}, \overline{10}, \overline{17}, \overline{24}, \ldots, \overline{n-6}\} \cup \{\overline{6}, \overline{13}, \overline{20}, \overline{27}, \ldots, \overline{n-3}\}$

which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq |X_2| = 3\lceil \frac{n}{7} \rceil - 2$.

For $n \equiv 3 \pmod 7$, we obtain that X_3 is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})$ where $X_3 = \{\overline{1}, \overline{8}, \overline{15}, \overline{22}, \dots, \overline{n-2}\} \cup \{\overline{3}, \overline{10}, \overline{17}, \overline{24}, \dots, \overline{n-7}\} \cup \{\overline{6}, \overline{13}, \overline{20}, \overline{27}, \dots, \overline{n-4}\}$ which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq |X_3| = 3\lceil \frac{n}{7} \rceil - 2$.

For $n \equiv 4 \pmod 7$, we obtain that X_4 is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})$ where $X_4 = \{\overline{1}, \overline{8}, \overline{15}, \overline{22}, \dots, \overline{n-3}\} \cup \{\overline{3}, \overline{10}, \overline{17}, \overline{24}, \dots, \overline{n-1}\} \cup \{\overline{6}, \overline{13}, \overline{20}, \overline{27}, \dots, \overline{n-5}\}$ which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq |X_4| = 3\lceil \frac{n}{7} \rceil - 1$.

For $n \equiv 5 \pmod 7$, we obtain that X_5 is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})$ where $X_5 = \{\overline{1}, \overline{8}, \overline{15}, \overline{22}, \dots, \overline{n-4}\} \cup \{\overline{3}, \overline{10}, \overline{17}, \overline{24}, \dots, \overline{n-2}\} \cup \{\overline{6}, \overline{13}, \overline{20}, \overline{27}, \dots, \overline{n-6}\}$ which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq |X_5| = 3\lceil \frac{n}{7} \rceil - 1$.

For $n \equiv 6 \pmod 7$, we obtain that X_6 is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})$ where $X_6 = \{\overline{1}, \overline{8}, \overline{15}, \overline{22}, \dots, \overline{n-5}\} \cup \{\overline{3}, \overline{10}, \overline{17}, \overline{24}, \dots, \overline{n-3}\} \cup \{\overline{6}, \overline{13}, \overline{20}, \overline{27}, \dots, \overline{n}\}$ which implies that $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{4}\})) \leq |X_6| = 3\lceil \frac{n}{7} \rceil$. □

Proposition 4.1.12. *Let $n \geq 5$ be a positive integer. Then $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{5}\})) \leq \lceil \frac{n}{3} \rceil + 1$.*

Proof. Let $n \geq 5$ be a positive integer.

For $n \equiv 0 \pmod 3$, we obtain that $X_0 = \{\overline{1}, \overline{2}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \dots, \overline{n-2}\}$ is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{5}\})$ which leads to $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{5}\})) \leq |X_0| = \lceil \frac{n}{3} \rceil + 1$.

For $n \equiv 1 \pmod 3$, we obtain that $X_1 = \{\overline{1}, \overline{2}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \dots, \overline{n}\}$ is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{5}\})$ which leads to $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{5}\})) \leq |X_1| = \lceil \frac{n}{3} \rceil + 1$.

For $n \equiv 2 \pmod 3$, we obtain that $X_2 = \{\overline{1}, \overline{2}, \overline{4}, \overline{7}, \overline{10}, \overline{13}, \dots, \overline{n-1}\}$ is a dominating set of $\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{5}\})$ which leads to $\gamma(\mathrm{Cay}(\mathbb{Z}_n, \{\overline{1}, \overline{5}\})) \leq |X_2| = \lceil \frac{n}{3} \rceil + 1$. □

Since a Cayley digraph of a left group can be considered as the disjoint union of Cayley digraphs of a group as shown in Lemma 2.2.5, we can directly obtain some results for the domination number of a Cayley digraph $\mathrm{Cay}(\mathbb{Z}_n \times L, A)$ of a left group $\mathbb{Z}_n \times L$ with a connection set A as follows.

Theorem 4.1.13. *Let $n \geq 2$ be a positive integer and A a nonempty subset of $G \times L$. If $p_1(A) = \{\overline{1}, \overline{2}\}$, then $\gamma(\mathrm{Cay}(\mathbb{Z}_n \times L, A)) = |L|\lceil \frac{n}{3} \rceil$.*

Theorem 4.1.14. *Let $n \geq 3$ be a positive integer and A a nonempty subset of $G \times L$. If* $p_1(A) = \{\overline{1}, \overline{3}\}$, *then* $\gamma(\mathrm{Cay}(\mathbb{Z}_n \times L, A)) = \begin{cases} |L|(2\lceil \frac{n}{5} \rceil - 1) & \text{if } n \equiv 1, 2 \pmod 5, \\ 2|L|\lceil \frac{n}{5} \rceil & \text{if } n \equiv 0, 3, 4 \pmod 5. \end{cases}$

Theorem 4.1.15. *Let $n \geq 4$ be a positive integer and A a nonempty subset of $G \times L$. If $p_1(A) = \{\overline{1}, \overline{4}\}$, then*
$$\gamma(\mathrm{Cay}(\mathbb{Z}_n \times L, A)) \leq \begin{cases} 3|L|\lceil \frac{n}{7} \rceil & \text{if } n \equiv 0, 6 \pmod 7, \\ |L|(3\lceil \frac{n}{7} \rceil - 1) & \text{if } n \equiv 4, 5 \pmod 7, \\ |L|(3\lceil \frac{n}{7} \rceil - 2) & \text{if } n \equiv 1, 2, 3 \pmod 7. \end{cases}$$

Theorem 4.1.16. *Let $n \geq 5$ be a positive integer and A a nonempty subset of $G \times L$. If $p_1(A) = \{\overline{1}, \overline{5}\}$, then $\gamma(\mathrm{Cay}(\mathbb{Z}_n \times L, A)) \leq |L|(\lceil \frac{n}{3} \rceil + 1)$.*

Now, we study the domination number of a Cayley digraph Λ of a right group $G \times R$ relative to the appropriate connection set A. We start with the following theorem which describes the domination number of Cayley digraphs of right groups with any connection set A where $|p_2(A)| \neq |R|$.

Theorem 4.1.17. *Let $G \times R$ be a right group and A a connection set of Λ. If $|p_2(A)| \neq |R|$, then $\gamma(\Lambda) = (|R| - |p_2(A)|) \times |G|$.*

Proof. Suppose that $|p_2(A)| \neq |R|$, we have $|p_2(A)| < |R|$. Let $Y = \{(x, a) \in G \times R : a \notin p_2(A)\}$. We will show that Y is a dominating set of Λ. Let $(b, c) \in (G \times R) \setminus Y$. Then $b \in G$ and $c \in p_2(A)$, that is, there exists $d \in p_1(A) \subseteq G$ such that $(d, c) \in A$. Since G is a group, there exists $y \in G$ such that $b = yd$. From $|R| > |p_2(A)|$, we get that there exists $r \in R \setminus p_2(A)$ which leads to $(y, r) \in Y$. Thus $(b, c) = (yd, c) = (y, r)(d, c)$. Therefore, Y is a dominating set of Λ. Hence $\gamma(\Lambda) \leq |Y| = (|R| - |p_2(A)|) \times |G|$. Now, we assume to the contrary that $\gamma(\Lambda) < (|R| - |p_2(A)|) \times |G|$. Let $X \subseteq G \times R$ be a dominating set of Λ such that X is a γ−set of Λ, that is, $|X| = \gamma(D) < (|R| - |p_2(A)|) \times |G|$. We have

$$\begin{aligned} |(G \times R) \setminus X| &= |G \times R| - |X| \\ &> |G||R| - [(|R| - |p_2(A)|) \times |G|] \\ &= |G||R| - |G||R| + |G||p_2(A)| \\ &= |G||p_2(A)| \\ &= |G \times p_2(A)|. \end{aligned}$$

Thus there exists at least one element $(a, b) \in ((G \times R) \setminus X) \setminus (G \times p_2(A))$, that is, $(a, b) \in (G \times R) \setminus X$ and $(a, b) \notin G \times p_2(A)$. Since $a \in G$, we obtain that $b \notin p_2(A)$. From $(a, b) \in (G \times R) \setminus X$ and X is the dominating set of Λ, there exists $(x, y) \in X$ such that $((x, y), (a, b)) \in E(\Lambda)$. Thus $(a, b) = (x, y)(c, d) = (xc, yd) = (xc, d)$ for some $(c, d) \in A$. We conclude that $b = d \in p_2(A)$ which is a contradiction. Consequently, $\gamma(\Lambda) \not< (|R| - |p_2(A)|) \times |G|$, that is, $\gamma(\Lambda) = (|R| - |p_2(A)|) \times |G|$, as required. □

The next theorem gives the bounds of the domination number in Cayley digraphs of right groups with any connection set A where $|p_2(A)| = |R|$.

Theorem 4.1.18. *Let $G \times R$ be a right group and A a connection set of Λ. If $|p_2(A)| = |R|$, then $\frac{|G||R|}{|A|+1} \leq \gamma(\Lambda) \leq |G|$.*

Proof. Assume that $|p_2(A)| = |R|$. We first prove the right inequality, that is, $\gamma(\Lambda) \leq |G|$. For each $r \in R$, let $Y = \{(x, r) : x \in G\} = G \times \{r\}$. We will show that Y is a dominating set of Λ. Let $(a, b) \in (G \times R) \setminus Y$. Then $a \in G$ and $b \in R$ such that $b \neq r$. Since $|p_2(A)| = |R|$ and $p_2(A) \subseteq R$, we get that $R = p_2(A)$ and then $b \in p_2(A)$. Thus there exists $c \in p_1(A) \subseteq G$ such that $(c, b) \in A$. Since G is a group, there exists $g \in G$ such that $a = gc$. We obtain that $(a, b) = (gc, b) = (g, r)(c, b)$ where $(g, r) \in Y$. Hence Y is the dominating set of Λ. Therefore, $\gamma(\Lambda) \leq |Y| = |G \times \{r\}| = |G|$.

Now, we will prove the left inequality. Let X be the dominating set of Λ such that X is a $\gamma-$set of Λ, that is, $|X| = \gamma(\Lambda)$. Then for each $(a, b) \in (G \times R) \setminus X$, we get that $(a, b) = (x, y)(s, t)$ for some $(x, y) \in X$ and $(s, t) \in A$ which implies that $(G \times R) \setminus X \subseteq XA$. Hence $|(G \times R) \setminus X| \leq |XA|$. Since every element of X has the same out-degree $|A|$, we can easily obtain that

$$\gamma(\Lambda)|A| = |X||A| \geq |XA| \geq |(G \times R) \setminus X| = |G \times R| - |X| = |G||R| - \gamma(\Lambda).$$

Then $\gamma(\Lambda)|A| \geq |G||R| - \gamma(D)$ which leads to $|G||R| \leq \gamma(\Lambda)|A| + \gamma(\Lambda) = \gamma(\Lambda)(|A| + 1)$. So we can conclude that $\gamma(\Lambda) \geq \frac{|G||R|}{|A|+1}$. □

The following example illustrates the sharpness of those bounds which are stated in Theorem 4.1.18.

Example 4.1.19. Let $\mathbb{Z}_3 \times R$ be a right group where $\mathbb{Z}_3$ is a group of integers modulo 3 under the addition and $R = \{r_1, r_2\}$ is a right zero semigroup.

(1). Consider the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_3 \times R, \{(\overline{2}, r_1), (\overline{2}, r_2)\})$ in Figure 4.3.
We have $X = \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{2}, r_1)\}$ is a $\gamma-$set of $\mathrm{Cay}(\mathbb{Z}_3 \times R, \{(\overline{2}, r_1), (\overline{2}, r_2)\})$ and then $\gamma(\mathrm{Cay}(\mathbb{Z}_3 \times R, \{(\overline{2}, r_1), (\overline{2}, r_2)\})) = |X| = 3 = |\mathbb{Z}_3|$.
Similarly, $\gamma(\mathrm{Cay}(\mathbb{Z}_n \times R, \{(\overline{2}, r_1), (\overline{2}, r_2)\})) = |\mathbb{Z}_n|$ where $n \in \mathbb{N}$.

(2). Consider the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_4 \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{1}, r_2)\})$ in Figure 4.4.
We have $Y = \{(\overline{0}, r_2), (\overline{2}, r_2)\}$ is a $\gamma-$set of $\mathrm{Cay}(\mathbb{Z}_4 \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{1}, r_2)\})$ and $\gamma(\mathrm{Cay}(\mathbb{Z}_4 \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{1}, r_2)\})) = |Y| = 2 = \frac{|\mathbb{Z}_4 \times R|}{|\{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{1}, r_2)\}|+1}$.

Similarly, we also obtain that $\gamma(\text{Cay}(\mathbb{Z}_{2k} \times R, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})) = k = \frac{4k}{4}$ $= \frac{|\mathbb{Z}_{2k} \times R|}{|\{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\}|+1}$ with a $\gamma-$set $\{(\bar{0}, r_2), (\bar{2}, r_2), (\bar{4}, r_2), \ldots, (\overline{2k-2}, r_2)\}$ where $k \in \mathbb{N}$.

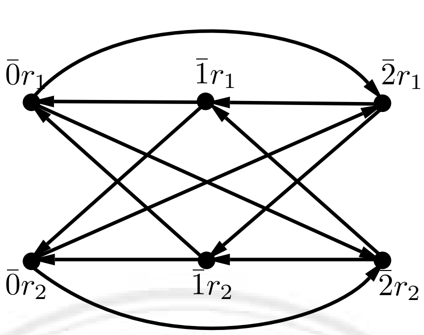

Figure 4.3: $\text{Cay}(\mathbb{Z}_3 \times R, \{(\bar{2}, r_1), (\bar{2}, r_2)\})$.

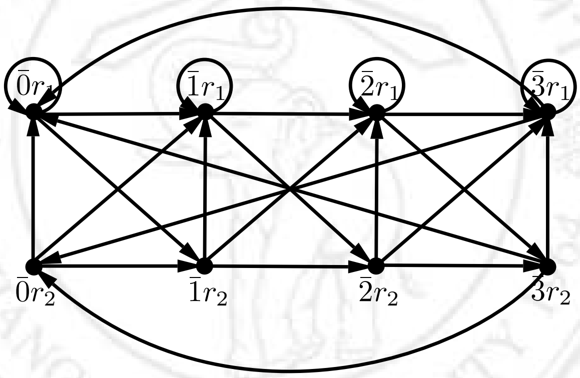

Figure 4.4: $\text{Cay}(\mathbb{Z}_4 \times R, \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{1}, r_2)\})$.

The following theorems present the value for the domination number of Cayley digraphs of right groups according to the specific connection sets. We begin with two lemmas which are referred in the proof of theorems.

Lemma 4.1.20. *Let $G \times R$ be a right group and A a connection set of Λ such that $p_1(A) = G$, $p_2(A) = R$, and $|A| = |R|$. For each $(x_1, r_1), (x_2, r_2) \in G \times R$, if $(x_1, r_1)(y_1, s_1) = (x_2, r_2)(y_2, s_2)$ for some $(y_1, s_1), (y_2, s_2) \in A$, then $x_1 = x_2$.*

Proof. Let $(x_1, r_1), (x_2, r_2) \in G \times R$ be such that $(x_1, r_1)(y_1, s_1) = (x_2, r_2)(y_2, s_2)$ for some $(y_1, s_1), (y_2, s_2) \in A$. Thus $(x_1y_1, r_1s_1) = (x_2y_2, r_2s_2)$, that is, $(x_1y_1, s_1) = (x_2y_2, s_2)$. Then $x_1y_1 = x_2y_2$ and $s_1 = s_2$. Since $p_1(A) = G$, $p_2(A) = R$, and $|A| = |R|$, these imply that $y_1 = y_2$. From $x_1y_1 = x_2y_2$ where x_1, x_2, y_1, y_2 are elements of a group G and $y_1 = y_2$, we can conclude that $x_1 = x_2$ by the cancellation law. □

Lemma 4.1.21. *Let $G \times R$ be a right group and A a connection set of Λ such that $A = \{a\} \times R$ where $a \in G$. Let Y be a dominating set of Λ. If there exists $x \in G$ such that $x \notin p_1(Y)$, then $(xa, r) \in Y$ for all $r \in R$.*

Proof. Let Y be a dominating set of Λ. Suppose that there exists $x \in G$ such that $x \notin p_1(Y)$ and assume to the contrary that there exists $r \in R$ such that $(xa, r) \notin Y$. Since Y is a dominating set of Λ, there exists $(y, r') \in Y$ such that $((y, r'), (xa, r)) \in E(\Lambda)$, that is, $(xa, r) = (y, r')(a, r)$ where $(a, r) \in A$. Hence $xa = ya$, it follows that $x = y \in p_1(Y)$ which contradicts to our supposition. Therefore, $(xa, r) \in Y$ for all $r \in R$. □

Theorem 4.1.22. *Let $G \times R$ be a right group and A a connection set of Λ such that $p_1(A) = G$, $p_2(A) = R$, and $|A| = |R|$. Then $\gamma(\Lambda) = |G|$.*

Proof. Assume that the conditions hold. Since $|p_2(A)| = |R|$, we obtain that $\gamma(\Lambda) \leq |G|$ by Theorem 4.1.18. Now, suppose that there exists a dominating set Y such that $|Y| < |G|$. Then there exists $g \in G$ such that $g \notin p_1(Y)$. We first prove that for each $r \in R$, $(g, r)A \subseteq Y$. Let $r \in R$ and $(x, y) \in (g, r)A$. Then $(x, y) = (g, r)(g_1, r_1)$ for some $(g_1, r_1) \in A$. If $(x, y) \notin Y$, then there exists $(g', r') \in Y$ such that $(x, y) = (g', r')(g_2, r_2)$ for some $(g_2, r_2) \in A$ since Y is a dominating set of Λ. Thus $(g, r)(g_1, r_1) = (x, y) = (g', r')(g_2, r_2)$ where $(g_1, r_1), (g_2, r_2) \in A$. By Lemma 4.1.20, we can conclude that $g = g' \in p_1(Y)$ which is a contradiction. Hence $(x, y) \in Y$ which leads to $(g, r)A \subseteq Y$. Since $p_1(A) = G$, we obtain that the identity element e of G lies in $p_1(A)$. Then there exists $s \in p_2(A)$ such that $(e, s) \in A$ and $(g, s) = (g, r)(e, s) \in (g, r)A \subseteq Y$. Whence, $g \in p_1(Y)$ which contradicts to the above supposition. Therefore, $\gamma(\Lambda) = |G|$. □

Theorem 4.1.23. *Let $G \times R$ be a right group and A a connection set of Λ such that $A = \{a\} \times R$ where $a \in G$. Then $\gamma(\Lambda) = |G|$.*

Proof. Let $G \times R$ be a right group and A a connection set of Λ such that $A = \{a\} \times R$ where $a \in G$. Then $|p_2(A)| = |R|$. By Theorem 4.1.18, we obtain that $\gamma(\Lambda) \leq |G|$. Assume that there exists a dominating set Y of Λ such that $|Y| < |G|$. Then there exists $x \in G$ such that $x \notin p_1(Y)$. Let $U = \{u \in G : u \notin p_1(Y)\}$. Assume that $|U| = k$ for some $k \in \mathbb{N}$ in which $1 \leq k \leq |G| - 1$. For each $u \in U$, we obtain by Lemma 4.1.21 that $(ua, r) \in Y$ for all $r \in R$. Hence there exists at least one element $q \in p_1(Y)$ such that $(q, r) \in Y$ for all $r \in R$. Let $V = \{v \in p_1(Y) : (v, r) \in Y$ for all $r \in R\}$. Assume that

$|V| = l$ for some $l \in \mathbb{N}$ such that $1 \leq l \leq |G| - k$. By Lemma 4.1.21 again, we get that $|Y| \geq |R|l + [(|G| - k) - l] + (k - l) = |R|l + |G| - 2l = |G| + (|R| - 2)l$. Since $|R| \geq 2$, we obtain that $|Y| \geq |G| + (|R| - 2)l \geq |G|$, a contradiction. Therefore, $\gamma(\Lambda) = |G|$, as required. □

Theorem 4.1.24. *Let $G \times R$ be a right group and $A = K \times R$ a connection set of Λ where K is any subgroup of G. Then $\gamma(\Lambda) = \frac{|G|}{|K|}$.*

Proof. Let $G \times R$ be a right group and $A = K \times R$ a connection set of Λ where K is a subgroup of a group G. Consider the set of all left cosets of K in G, $G/K = \{g_1K, g_2K, \ldots, g_tK\}$ for some $t \in \mathbb{N}$, we obtain that the index of K in G equals t, that is, $[G : K] = t$. Let $I = \{1, 2, \ldots, t\}$ be an index set. By Lemma 2.2.7, we have $(G \times R)/\langle A \rangle = \{g_iK \times R : i \in I\}$ such that $G \times R = \dot{\bigcup_{i \in I}}(g_iK \times R)$ and $\Lambda = \dot{\bigcup_{i \in I}}((g_iK \times R), E_i)$ where $((g_iK \times R), E_i)$ is a strong subdigraph of Λ in which $((g_iK \times R), E_i) \cong \mathrm{Cay}(\langle A \rangle, A)$ for all $i \in I$. Consequently,

$$\gamma(\Lambda) = \gamma\left(\dot{\bigcup_{i \in I}}((g_iK \times R), E_i)\right) = \sum_{i=1}^{t} \gamma((g_iK \times R), E_i) = t[\gamma(\mathrm{Cay}(\langle A \rangle, A))].$$

By Lemma 2.2.8, we can conclude that $\langle A \rangle = \langle p_1(A) \rangle \times p_2(A) = \langle K \rangle \times R = K \times R = A$. In this case, we can prove that $\gamma(\mathrm{Cay}(\langle A \rangle, A)) = 1$ since $\langle A \rangle = A$. So we can conclude that $\gamma(D) = t = [G : K] = \frac{|G|}{|K|}$ which completes the proof. □

4.2 Total Domination Number

For the total domination number of a Cayley digraph Δ of a rectangular group $G \times L \times R$ with a connection set A, we obtain that the total domination number of Δ will exist when $p_3(A) = R$. If we define $\overline{A}$ as in Theorem 4.1.1, then we can say that the total domination number of Δ will exist when $p_2(\overline{A}) = R$ which is considered in Cayley digraphs of right groups. So we will prove this condition in the part of the total domination number of Cayley digraphs of right groups (Theorem 4.2.11).

Now, we will present the result for the total domination number of a Cayley digraph of a rectangular group in the term of the total domination number of a Cayley digraph of a right group with a given connection set.

Theorem 4.2.1. *Let $G \times L \times R$ be a rectangular group and A a connection set of Δ. If $\overline{A} = \{(g, r) \in G \times R : (g, l, r) \in A$ for some $l \in L\}$ such that $p_2(\overline{A}) = R$, then $\gamma_t(\Delta) = |L| \cdot \gamma_t(\mathrm{Cay}(G \times R, \overline{A}))$.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.1.1. □

Next, we give some results of the total domination number of Cayley digraphs of left groups with their connection sets. We start with the lemma which gives the condition for the existence of total dominating sets in Cayley digraphs of left groups.

Lemma 4.2.2. *Let $G \times L$ be a left group and A a connection set of Γ. Then the total dominating set of Γ exists if and only if $A \neq \emptyset$.*

Proof. Suppose that the total dominating set of D exists, say T. By the definition of T, we obtain that for each $(g, l) \in G \times L$, (g, l) is dominated by (g_1, l_1) for some $(g_1, l_1) \in T$, that is, $((g_1, l_1), (g, l)) \in E(D)$. Then $(g, l) = (g_1, l_1)(a_1, l_2)$ where $(a_1, l_2) \in A$ which implies that $A \neq \emptyset$.

Conversely, assume that the connection set $A \neq \emptyset$, that is, there exists $(a, l) \in A$. Hence for each $(g_1, l_1) \in G \times L$, we obtain that $(g_1, l_1) = (g_1a^{-1}, l_1)(a, l)$ in which $(g_1a^{-1}, l_1) \in G \times L$. Thus (g_1, l_1) is dominated by (g_1a^{-1}, l_1) in $G \times L$. If we take $T = G \times L$, then we can conclude that T is a total dominating set of Γ, that is, the total dominating set of Γ always exists when $A \neq \emptyset$. □

The following result gives us the total domination number of a Cayley digraph Γ of a left group $G \times L$ with a connection set A in the term of the total domination number of a Cayley digraph of its subgroup.

Theorem 4.2.3. *Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$ for some $k \in \mathbb{N}$. Then $\gamma_t(\Gamma) = k \cdot |L| \cdot \gamma_t(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$ where A is a connection set of Γ.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.1.2. □

Proposition 4.2.4. *Let $S = D_n \times L$ be a left group and A a nonempty subset of S such that the identity element $e \notin p_1(A)$ and for each $x \in p_1(A)$, x^{-1} must belong to $p_1(A)$. Let $n \geq 3$ be an integer, $c = \lfloor \frac{n-1}{2} \rfloor$ and k, t be integers such that $1 \leq k \leq c$ and $1 \leq t \leq n$. Let $p_1(A) = \{r^{a_1}, r^{a_2}, \ldots, r^{a_k}, r^{n-a_k}, r^{n-a_{k-1}}, \ldots, r^{n-a_1}, sr^{b_1}, sr^{b_2}, \ldots, sr^{b_t}\} \subseteq D_n$.*
If $d = \max\{a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, b_1, b_2 - b_1, b_3 - b_2, \ldots, b_t - b_{t-1}\}$, then $\gamma_t(\mathrm{Cay}(S, A)) \leq 2d|L|\lceil \frac{n}{d+2a_k} \rceil$.

Proof. This proposition is true by Lemma 2.2.5 and Lemma 2.3.4. □

Proposition 4.2.5. *Let $n \geq 3$ be an odd integer and $c = \frac{n-1}{2}$. Let $S = \mathbb{Z}_n \times L$ be a left group and A a nonempty subset of S such that*

$$p_1(A) = \{c, n-c, c-1, n-(c-1), \ldots, c-(k-1), n-(c-(k-1))\}$$

for some $k \in \mathbb{N}$ in which $1 \leq k \leq c$. Then $\gamma_t(\mathrm{Cay}(S, A)) = |L|\lceil \frac{n}{2k} \rceil$.

Proof. This proposition follows from Lemma 2.2.5 and Lemma 2.3.5, directly. □

Proposition 4.2.6. *Let $n \geq 3$ be an even integer and $c = \lfloor \frac{n-1}{2} \rfloor$. Let $S = \mathbb{Z}_n \times L$ be a left group and A a nonempty subset of S such that*

$$p_1(A) = \{\tfrac{n}{2}, c, n-c, c-1, n-(c-1), \ldots, c-(k-1), n-(c-(k-1))\}$$

for some $k \in \mathbb{N}$ in which $1 \leq k \leq c$. Then $\gamma_t(\mathrm{Cay}(S, A)) = |L| \lceil \frac{n}{2k+1} \rceil$.

Proof. This proposition follows from Lemma 2.2.5 and Lemma 2.3.6, directly. □

Before we give the next lemmas, we will define some notations which are used in the proof. Let $I = [a, b]$ be an interval of consecutive integers x such that $a \leq x \leq b$. Furthermore, let (V, E) be a digraph and for each $u \in V$, let $N(u) = \{v \in V : (u, v) \in E\}$ be the set of all neighbours of a vertex u and $N(A) = \bigcup\limits_{a \in A} N(a)$ where A is a subset of V.

Lemma 4.2.7. *Let $n \geq 3$ be an odd integer. Let $m = \frac{n-1}{2}$ and k be a fixed integer such that $1 \leq k \leq m$. If $A = \{m, m-1, m-2, \ldots, m-(k-1)\} \subseteq \mathbb{Z}_n$, then $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) = \lceil \frac{n}{k} \rceil$.*

Proof. Assume that $A = \{m, m-1, m-2, \ldots, m-(k-1)\}$ and let $l = \lceil \frac{n}{k} \rceil$. Since every vertex in $\mathbb{Z}_n$ has an out-degree k, from the definition of the total domination number, it follows that $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) \geq l$. Let $x = m + k + 1$ and

$$X_t = \{x, x+k, x+2k, \ldots, x+(l-1)k\}.$$

Note that $|X_t| = l$. Since $l = \lceil \frac{n}{k} \rceil$, we get that $n = (l-1)k + j$ for some $j \in \mathbb{N}$ with $1 \leq j \leq k$. Thus $V(\mathrm{Cay}(\mathbb{Z}_n, A))$ can be partitioned into l intervals as follows:

$$I_1 = [1, k],\ I_2 = [k+1, 2k],\ I_3 = [2k+1, 3k], \ldots, I_{l-1} = [(l-2)k+1, (l-1)k], \text{ and}$$
$$I_l = [(l-1)k+1, n].$$

Note that $|I_i| = k$ for all i with $1 \leq i \leq l-1$ and $1 \leq |I_l| \leq k$. For any $0 \leq i \leq l-2$, we have $x + ik \in X_t$ and $I_{i+1} = [ik+1, (i+1)k]$. Since A is a set of k consecutive integers with the least element $m-(k-1)$ and from $(x+ik) + (m-(k-1)) \equiv ik+1 \pmod{n}$, we obtain that $N(x+ik) = I_{i+1}$. Therefore,

$$(x+(l-1)k) + m - (k-1) \equiv (l-1)k + 1 \pmod{n} \text{ and so } I_l \subseteq N(x+(l-1)k).$$

Consequently,

$$\begin{aligned} V(\mathrm{Cay}(\mathbb{Z}_n, A)) &= I_1 \cup I_2 \cup \cdots \cup I_{l-1} \cup I_l \\ &\subseteq N(x) \cup N(x+k) \cup \cdots \cup N(x+(l-2)k) \cup N(x+(l-1)k) \\ &= \bigcup_{y \in X_t} N(y) = N(X_t). \end{aligned}$$

Thus X_t is a total dominating set of $\mathrm{Cay}(\mathbb{Z}_n, A)$. Hence $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) \leq |X_t| = l$. So we can conclude that $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) = l = \lceil \frac{n}{k} \rceil$, as desired. $\square$

Lemma 4.2.8. *Let $n \geq 3$ be an even integer. Let $m = \lfloor \frac{n-1}{2} \rfloor$ and k be a fixed integer such that $1 \leq k \leq m$. If $A = \{\frac{n}{2}, m, m-1, \ldots, m-(k-1)\} \subseteq \mathbb{Z}_n$, then $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) = \lceil \frac{n}{k+1} \rceil$.*

Proof. Suppose that $A = \{\frac{n}{2}, m, m-1, \ldots, m-(k-1)\}$. Then $|A| = k+1$ and let $l = \lceil \frac{n}{k+1} \rceil$. Since every vertex of $\mathrm{Cay}(\mathbb{Z}_n, A)$ has an out-degree $k+1$, we also have $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) \geq l$. Let $x = m + k + 2$ and $X_t = \{x, x + (k+1), x + 2(k+1), \ldots, x + (l-1)(k+1)\}$. By partitioning the set of all vertices of $\mathrm{Cay}(\mathbb{Z}_n, A)$ into l intervals as follows:

$$I_1 = [1, k+1], I_2 = [(k+1)+1, 2(k+1)], \ldots, I_{l-1} = [(l-2)(k+1)+1, (l-1)(k+1)],$$
$$\text{and } I_l = [(l-1)(k+1)+1, n],$$

we can prove the remaining part of this lemma by applying the proof of the previous lemma, similarly. Hence we can get the similar result of the previous proof, that is, $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) \leq |X_t| = l$. Therefore, $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n, A)) = l = \lceil \frac{n}{k+1} \rceil$. $\square$

Now, we apply the above two lemmas to obtain the results for the total domination number of a Cayley digraph of a left group $\mathbb{Z}_n \times L$ with an according connection set.

Theorem 4.2.9. *Let $n \geq 3$ be an odd integer. Let $c = \frac{n-1}{2}$ and k be a fixed integer such that $1 \leq k \leq c$. Let $S = \mathbb{Z}_n \times L$ be a left group and A a nonempty subset of S. If $p_1(A) = \{c, c-1, c-2, \ldots, c-(k-1)\}$, then $\gamma_t(\mathrm{Cay}(S, A)) = |L| \lceil \frac{n}{k} \rceil$.*

Proof. This theorem is a direct result from Lemma 2.2.5 and Lemma 4.2.7. $\square$

Theorem 4.2.10. *Let $n \geq 3$ be an even integer. Let $c = \lfloor \frac{n-1}{2} \rfloor$ and k be a fixed integer such that $1 \leq k \leq c$. Let $S = \mathbb{Z}_n \times L$ be a left group and A a nonempty subset of S. If $p_1(A) = \{\frac{n}{2}, c, c-1, \ldots, c-(k-1)\}$, then $\gamma_t(\mathrm{Cay}(S, A)) = |L| \lceil \frac{n}{k+1} \rceil$.*

Proof. This theorem follows from Lemma 2.2.5 and Lemma 4.2.8, directly. $\square$

Now, we present some results of the total domination number of Cayley digraphs of right groups with their connection sets. The following theorem gives the necessary and sufficient conditions for the existence of the total dominating set of a Cayley digraph Λ of a right group $G \times R$ with a connection set A.

Theorem 4.2.11. *Let $G \times R$ be a right group and A a connection set of Λ. Then the total dominating set of Λ exists if and only if $p_2(A) = R$.*

Proof. We first prove the necessary condition by assuming that the total dominating set of Λ exists, say T. We will show that $p_2(A) = R$. By the definition of the connection set A, we have known that $p_2(A) \subseteq R$. Let $r \in R$. Then for each $a \in G$, we get that (a, r) is dominated by a vertex (x, y) for some $(x, y) \in T$ since T is the total dominating set of Λ. Thus there exists $(a', r') \in A$ such that $(a, r) = (x, y)(a', r') = (xa', yr') = (xa', r')$ which implies that $r = r'$, that is, $r \in p_2(A)$. Therefore, $p_2(A) = R$.

Conversely, we prove the sufficient condition by supposing that $p_2(A) = R$. We will prove that every vertex has an in-degree in Λ. Let $(g, r) \in G \times R$. Then $r \in R = p_2(A)$. Thus there exists $a \in p_1(A)$ such that $(a, r) \in A$. We obtain that

$$((ga^{-1}, r'), (g, r)) = ((ga^{-1}, r'), (ga^{-1}, r')(a, r)) \in E(\Lambda),$$

that is, (g, r) is dominated by (ga^{-1}, r'). So we can conclude that every vertex of Λ always has an in-degree in Λ. If we take $T = V(\Lambda) = G \times R$, then we can see that T is a total dominating set of Λ since for each $(x, y) \in G \times R$, (x, y) is dominated by some vertices in T. Hence the total dominating set of Λ always exists if $p_2(A) = R$. □

Consequently in this part, we need to consider the connection set A of Λ in the case where $p_2(A) = R$. Next, we will show a lower bound and an upper bound of the total domination number of a Cayley digraph of a right group with a given connection set.

Theorem 4.2.12. *Let $G \times R$ be a right group and A a connection set of Λ such that $p_2(A) = R$. Then $\frac{|G||R|}{|A|} \leq \gamma_t(\Lambda) \leq |G|$.*

Proof. Let A be a connection set of Λ such that $p_2(A) = R$. We have known that the total dominating set of Λ exists by Theorem 4.2.11. For each $r \in R$, we let

$$T = \{(g, r) : g \in G\} = G \times \{r\}.$$

We will show that T is a total dominating set of Λ. Let $(x, y) \in S = G \times R$. Since $p_2(A) = R$, we get that $y \in p_2(A)$ which implies that there exists $z \in p_1(A)$ such that $(z, y) \in A$. Since G is a group and $x, z \in G$, we obtain that $x = hz$ for some $h \in G$. Thus there exists $(h, r) \in T$ such that $(x, y) = (hz, y) = (h, r)(z, y)$. Hence (x, y) is dominated by the vertex (h, r) in T. We can conclude that T is the total dominating set of Λ which leads to the fact that

$$\gamma_t(\Lambda) \leq |T| = |G \times \{r\}| = |G|.$$

Next, we will show that $\gamma_t(\Lambda) \geq \frac{|G||R|}{|A|}$. Assume to the contrary that there exists a total dominating set T' such that $|T'| < \frac{|G||R|}{|A|}$. Thus $|T'A| \leq |T'||A| < |G||R| = |G \times R|$

which implies that there exists at least one element $(p, q) \in G \times R$ but $(p, q) \notin T'A$. Hence there is no an element in T' which dominates (p, q), this contradicts to the property of the total dominating set T'. Consequently, $\gamma_t(\Lambda) \geq \frac{|G||R|}{|A|}$, as required. □

In the following example, we illustrate the sharpness of those bounds which are given in Theorem 4.2.12.

Example 4.2.13. Let $\mathbb{Z}_3 \times R$ be a right group where $\mathbb{Z}_3$ is a group of integers modulo 3 under the addition and $R = \{r_1, r_2\}$ is a right zero semigroup.

(1). Consider the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_3 \times R, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})$.

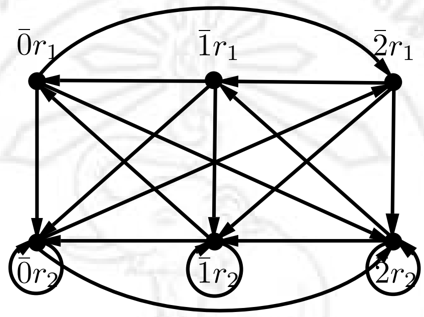

Figure 4.5: $\mathrm{Cay}(\mathbb{Z}_3 \times R, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})$.

We obtain that $X = \{(\bar{0}, r_1), (\bar{1}, r_1), (\bar{2}, r_1)\}$ is a γ_t-set of the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_3 \times R, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})$ and then

$$\gamma_t(\mathrm{Cay}(\mathbb{Z}_3 \times R, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})) = |X| = 3 = |\mathbb{Z}_3|.$$

Similarly, we can obtain that $\gamma_t(\mathrm{Cay}(\mathbb{Z}_n \times R, \{(\bar{2}, r_1), (\bar{0}, r_2), (\bar{2}, r_2)\})) = |\mathbb{Z}_n|$ with a γ_t-set $\mathbb{Z}_n \times \{r_1\}$ where $n \in \mathbb{N}$.

(2). Consider $\text{Cay}(\mathbb{Z}_3 \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{2}, r_1), (\overline{0}, r_2), (\overline{1}, r_2), (\overline{2}, r_2)\})$.

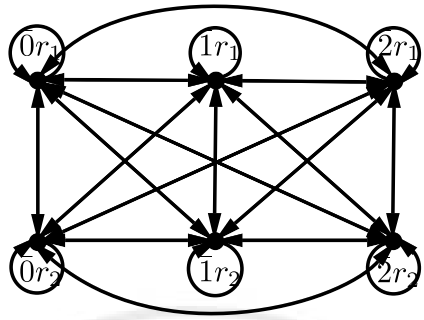

Figure 4.6: $\text{Cay}(\mathbb{Z}_3 \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{2}, r_1), (\overline{0}, r_2), (\overline{1}, r_2), (\overline{2}, r_2)\})$.

Then $Y = \{(\overline{0}, r_1)\}$ is a γ_t-set of $\text{Cay}(\mathbb{Z}_3 \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{2}, r_1), (\overline{0}, r_2), (\overline{1}, r_2), (\overline{2}, r_2)\})$ and hence $\gamma_t(\text{Cay}(\mathbb{Z}_3 \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{2}, r_1), (\overline{0}, r_2), (\overline{1}, r_2), (\overline{2}, r_2)\})) = |Y| = \frac{|\mathbb{Z}_3 \times R_2|}{|A|}$ where $A = \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{2}, r_1), (\overline{0}, r_2), (\overline{1}, r_2), (\overline{2}, r_2)\}$.

Similarly, we get that $\gamma_t(\text{Cay}(\mathbb{Z}_{3k} \times R, \{(\overline{0}, r_1), (\overline{1}, r_1), (\overline{2}, r_1), (\overline{0}, r_2), (\overline{1}, r_2), (\overline{2}, r_2)\})) = \frac{|\mathbb{Z}_{3k} \times R_2|}{|A|} = k$ with a γ_t-set $\{(\overline{0}, r_1), (\overline{3}, r_1), (\overline{6}, r_1), \ldots, (\overline{3k-3}, r_1)\}$ where $k \in \mathbb{N}$.