CHAPTER 5

Independence on Cayley Digraphs of Rectangular Groups

In this chapter, we propose some results about independence parameters consisting of the independence number, weakly independence number, dipath independence number, and weakly dipath independence number on Cayley digraphs of rectangular groups with respect to their connection sets. Furthermore, we study those independence parameters on Cayley digraphs of left groups and right groups which are semigroups that encode some structures of rectangular groups.

For a connection set A of a Cayley digraph, if $p_1(A)$ contains the identity e of a group G, then we can observe that every vertex of the digraph has a loop attaching to itself which does not affect to those independence parameters. Throughout this chapter, we thus suffice to consider the connection set A in which the identity $e \notin p_1(A)$.

5.1 Independence Number

This section presents some results of an independence number on Cayley digraphs of rectangular groups including left groups and right groups. Recall that, the digraph Δ denotes a Cayley digraph $\mathrm{Cay}(G \times L \times R, A)$ of a rectangular group $G \times L \times R$ with respect to a connection set A. In fact, the digraph Δ can be considered as the disjoint union of $|L|$ strong subdigraphs $(G \times \{\ell\} \times R, E_\ell)$ such that $(G \times \{\ell\} \times R, E_\ell)$ is isomorphic to $\mathrm{Cay}(G \times R, \overline{A})$ for all $\ell \in L$ where

$$\overline{A} = \{(a, \lambda) \in G \times R : (a, l, \lambda) \in A \text{ for some } l \in L\}.$$

So those results on Cayley digraphs of rectangular groups will depend on the results from corresponding Cayley digraphs of right groups, certainly. Some results of the parameter α of the digraph Δ are obtained as follows.

Theorem 5.1.1. *If I is an $\alpha-$set of Δ, then $I \cap (G \times \{\ell\} \times R)$ is an $\alpha-$set of the digraph $(G \times \{\ell\} \times R, E_\ell)$ for all $\ell \in L$.*

Proof. Let I be an $\alpha-$set of Δ and $\ell \in L$. We will show that $I \cap (G \times \{\ell\} \times R)$ is an $\alpha-$set of the digraph $(G \times \{\ell\} \times R, E_\ell)$. It is obvious that $I \cap (G \times \{\ell\} \times R)$ is a nonempty subset of $G \times \{\ell\} \times R$. Since $I \cap (G \times \{\ell\} \times R) \subseteq I$, we can conclude that $I \cap (G \times \{\ell\} \times R)$ is an independent set of $(G \times \{\ell\} \times R, E_\ell)$. Assume that there exists an independent set J of $(G \times \{\ell\} \times R, E_\ell)$ such that $|J| > |I \cap (G \times \{\ell\} \times R)|$. Therefore, $[I \setminus (G \times \{\ell\} \times R)] \cup J$ is an independent set of Δ and we also obtain that

$$\begin{aligned} |[I \setminus (G \times \{\ell\} \times R)] \cup J| &= |I \setminus (G \times \{\ell\} \times R)| + |J| \\ &> |I \setminus (G \times \{\ell\} \times R)| + |I \cap (G \times \{\ell\} \times R)| \\ &= |[I \setminus (G \times \{\ell\} \times R)] \cup [I \cap (G \times \{\ell\} \times R)]| \\ &= |I| \quad \text{which is a contradiction.} \end{aligned}$$

Consequently, our assertion is completely proved. $\square$

Theorem 5.1.2. *If T is an $\alpha-$set of* $\mathrm{Cay}(G \times R, \overline{A})$, *then* $\bigcup_{(t,\lambda)\in T} (\{t\} \times L \times \{\lambda\})$ *is an $\alpha-$set of Δ.*

Proof. Suppose that T is an $\alpha-$set of $\mathrm{Cay}(G \times R, \overline{A})$. Let $\bigcup_{(t,\lambda)\in T} (\{t\} \times L \times \{\lambda\})$ be denoted by K. We will prove that K is an $\alpha-$set of Δ. It is clear that K is a nonempty subset of $G \times L \times R$. We first show that K is independent in Δ. Assume that there exist $(t_1, l_1, \lambda_1), (t_2, l_2, \lambda_2) \in K$ such that they are not independent, that is,

$$((t_1, l_1, \lambda_1), (t_2, l_2, \lambda_2)) \in E(\Delta) \text{ or } ((t_2, l_2, \lambda_2), (t_1, l_1, \lambda_1)) \in E(\Delta).$$

Without loss of generality, we may suppose that $((t_1, l_1, \lambda_1), (t_2, l_2, \lambda_2)) \in E(D)$, that is, $(t_2, l_2, \lambda_2) = (t_1, l_1, \lambda_1)(a, l, \mu) = (t_1a, l_1, \mu)$ for some $(a, l, \mu) \in A$. Hence $(a, \mu) \in \overline{A}$ and $(t_2, \lambda_2) = (t_1a, \mu) = (t_1, \lambda_1)(a, \mu)$ which implies that $((t_1, \lambda_1), (t_2, \lambda_2)) \in E(\mathrm{Cay}(G \times R, \overline{A}))$ where $(t_1, \lambda_1), (t_2, \lambda_2) \in T$ which contradicts to the independence of T. Thus K is an independent set of Δ. We now assume that there exists an independent set M of Δ such that $\alpha(\Delta) = |M| > |K| = |\bigcup_{(t,\lambda)\in T} (\{t\} \times L \times \{\lambda\})| = |T||L|$. Then there exists $\ell \in L$ such that $|M \cap (G \times \{\ell\} \times R)| > |T|$. Since M is an $\alpha-$set of Δ, we obtain that $M \cap (G \times \{\ell\} \times R)$ is an $\alpha-$set of the digraph $(G \times \{\ell\} \times R, E_\ell)$ by Theorem 5.1.1. Thus $\alpha(G \times \{\ell\} \times R, E_\ell) = |M \cap (G \times \{\ell\} \times R)|$. Since we have known that $(G \times \{\ell\} \times R, E_\ell) \cong \mathrm{Cay}(G \times R, \overline{A})$, we can conclude that

$$\alpha(\mathrm{Cay}(G \times R, \overline{A})) = \alpha(G \times \{\ell\} \times R, E_\ell) = |M \cap (G \times \{\ell\} \times R)| > |T|$$

which is a contradiction because T is an $\alpha-$set of $\mathrm{Cay}(G \times R, \overline{A})$. Consequently, the set $\bigcup_{(t,\lambda)\in T} (\{t\} \times L \times \{\lambda\})$ is an $\alpha-$set of Δ, as required. $\square$

Next, we will show some results of an independence number of Cayley digraphs of left groups. Indeed, a Cayley digraph Γ of a left group $G \times L$ with a connection set A is the disjoint union of strong subdigraphs which each subdigraph is isomorphic to $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ as stated in Lemma 2.2.6. Thus we first need to consider the Cayley digraph $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ for extending the results to the Cayley digraphs of left groups with corresponding connection sets. Firstly, we show some facts of an independence number on Cayley digraphs of left groups.

Let A be a connection set of Γ. Thus $p_1(A)$ is a connection set of $\mathrm{Cay}(G, p_1(A))$. Let A^* be another connection set of a Cayley digraph Γ of a left group $G \times L$ such that

$$p_1(A^*) = p_1(A) \cup C \text{ where } C \subseteq \{c^{-1} : c \in p_1(A)\}.$$

Clearly, $p_1(A^*) \subseteq \langle p_1(A)\rangle$. We then obtain the following theorem.

Theorem 5.1.3. $\alpha(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = \alpha(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A^*)))$.

Proof. Let $H := \mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ and $H^* := \mathrm{Cay}(\langle p_1(A)\rangle, p_1(A^*))$ be Cayley digraphs of $\langle p_1(A)\rangle$ with connection sets $p_1(A)$ and $p_1(A^*)$, respectively. Since $p_1(A) \subseteq p_1(A^*)$, we get that H is a spanning subdigraph of H^* which easily implies that $\alpha(H^*) \leq \alpha(H)$. Suppose that $\alpha(H) = k$ with a corresponding $\alpha-$set $X = \{x_1, x_2, \ldots, x_k\}$ for some $k \in \mathbb{N}$. We will show that X is independent in H^*. Assume that there exist x_i and x_j in X such that they are not independent, that is, $(x_i, x_j) \in E(H^*)$ or $(x_j, x_i) \in E(H^*)$. Without loss of generality, we can take $(x_i, x_j) \in E(H^*)$. Hence $x_j = x_i a$ for some $a \in p_1(A^*)$.

If $a \in p_1(A)$, then $(x_i, x_j) \in E(H)$ which is a contradiction since x_i and x_j are independent in H.

If $a = c^{-1}$ for some $c \in p_1(A)$, then we have $x_j = x_i a = x_i c^{-1}$ which implies that $x_i = x_j c$, that is, $(x_j, x_i) \in E(H)$ which again contradicts to the property of the set X.

Therefore, X is an independent set of H^* which directly implies that $\alpha(H) = k = |X| \leq \alpha(H^*)$. So we can conclude that $\alpha(H) = \alpha(H^*)$. $\square$

The following lemma describes a lower bound and an upper bound of an independence number on Cayley digraphs of groups which is the useful result to obtain those bounds on Cayley digraphs of left groups.

Lemma 5.1.4. *Let* $\mathrm{Cay}(G, B)$ *be a Cayley digraph of a group* G *with a connection set* B. *Then* $\frac{|G|}{|\langle B\rangle|}\left\lceil \frac{|\langle B\rangle|}{2|B|+1}\right\rceil \leq \alpha(\mathrm{Cay}(G, B)) \leq \frac{|G|}{|\langle B\rangle|}\left\lfloor \frac{|\langle B\rangle|}{2}\right\rfloor$.

Proof. Let B be a connection set of $\mathrm{Cay}(G, B)$. It is the fact that $\mathrm{Cay}(G, B)$ is the disjoint union of $\frac{|G|}{|\langle B\rangle|}$ subdigraphs such that each subdigraph is isomorphic to $\mathrm{Cay}(\langle B\rangle, B)$, so we need to consider an independence number of $\mathrm{Cay}(\langle B\rangle, B)$. Suppose that $\alpha(\mathrm{Cay}(\langle B\rangle, B)) = k$ for some $k \in \mathbb{N}$. Since every vertex has an in-degree $|B|$ and an out-degree $|B|$, we get that $k(2|B|+1) \geq |\langle B\rangle|$ and whence $k \geq \left\lceil \frac{|\langle B\rangle|}{2|B|+1} \right\rceil$. Next, we suppose that there exists an $\alpha-$set X of $\mathrm{Cay}(\langle B\rangle, B)$ in which $|X| \geq \left\lfloor \frac{|\langle B\rangle|}{2} \right\rfloor + 1$. Then for each $b \in B$ and for all $x \in X$, we conclude that $xb \notin X$ by the independence of the set X. Hence $X \cap Xb = \emptyset$. Thus

$$\begin{aligned} |\langle B\rangle| \geq |X \cup Xb| = |X| + |Xb| &\geq \left\lfloor \tfrac{|\langle B\rangle|}{2} \right\rfloor + \left\lfloor \tfrac{|\langle B\rangle|}{2} \right\rfloor + 2 \\ &= \tfrac{|\langle B\rangle|-t}{2} + \tfrac{|\langle B\rangle|-t}{2} + 2 \quad \text{where } t \in \{0,1\} \\ &= \tfrac{2|\langle B\rangle|-2t+4}{2} \\ &= |\langle B\rangle| - t + 2 > |\langle B\rangle|. \end{aligned}$$

This gives a contradiction. By applying the fact mentioned in the beginning of the proof, we can conclude that $\alpha(\mathrm{Cay}(G, B))$ satisfies the lower bound and upper bound stated in the above assertion. □

Theorem 5.1.5. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. Then* $\frac{|G||L|}{|\langle p_1(A)\rangle|} \left\lceil \frac{|\langle p_1(A)\rangle|}{2|p_1(A)|+1} \right\rceil \leq \alpha(\Gamma) \leq \frac{|G||L|}{|\langle p_1(A)\rangle|} \left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor$.

Proof. According to Lemma 2.2.5, we obtain that Γ is the disjoint union of $|L|$ isomorphic subdigraphs such that each subdigraph is isomorphic to a Cayley digraph $\mathrm{Cay}(G, p_1(A))$ of a group G with a connection set $p_1(A)$. Thus the lower bound and upper bound of $\alpha(\Gamma)$ are directly obtained from Lemma 5.1.4. □

The following proposition illustrates the sharpness of the lower bound and upper bound of $\alpha(\Gamma)$ shown in Theorem 5.1.5. In order to show the proposition, we need to prescribe some notations as follows. Let A be a connection set of a Cayley digraph with a vertex set V and an arc set E. We define

$$A^{-1} = \{a^{-1} : a \in A\};\ N^+(x) = \{y \in V : (x, y) \in E\}; \text{ and } N^-(x) = \{y \in V : (y, x) \in E\}.$$

Proposition 5.1.6. *The bounds given in the above theorem are sharp.*

Proof. Let G be a finite cyclic group of odd order n with the generator g and A a connection set of Γ such that $p_1(A) = \{g, g^2, g^3, \ldots, g^{\frac{n-1}{2}}\}$. It is obvious that $\langle p_1(A)\rangle = G$ and $p_1(A) \cap [p_1(A)]^{-1} = \emptyset$ where $[p_1(A)]^{-1} = \{a^{-1} : a \in p_1(A)\}$. Then for each $x \in \langle p_1(A)\rangle$, we have $|N^+(x)| + |N^-(x)| = \frac{n-1}{2} + \frac{n-1}{2} = n-1$. Thus $\{x\}$ is an $\alpha-$set of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$. Therefore, $\alpha(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = 1 = \left\lceil \frac{|\langle p_1(A)\rangle|}{2|p_1(A)|+1} \right\rceil$ which leads to the lower bound of $\alpha(\Gamma)$ by applying Lemma 2.2.5, as desired.

Next, let A be a connection set of Γ in which $p_1(A) = \{a\}$ where a is not the identity of the group G and $|a| = k$ for some an odd number $k \in \mathbb{N}$. We will give the sharpness for the upper bound of an independence number of $\text{Cay}(\langle p_1(A)\rangle, p_1(A))$. Then we can easily investigate that the set $\{a, a^3, a^5, \ldots, a^{k-2}\}$ is an $\alpha-$set of $\text{Cay}(\langle p_1(A)\rangle, p_1(A))$, Consequently, $\alpha(\text{Cay}(\langle p_1(A)\rangle, p_1(A))) = \lfloor \frac{k}{2} \rfloor = \left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor$. Again by Lemma 2.2.5, we can conclude that $\alpha(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|} \left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor$ which reachs the required upper bound of $\alpha(\Gamma)$. Hence those bounds mentioned in Theorem 5.1.5 are certainly sharp. $\square$

In this part, we present some facts about the independence number of a Cayley digraph Λ of a right group $G \times R$ with respect to a connection set A. Throughout this part, we also focus on the connection set A in which the identity $e \notin p_1(A)$.

Theorem 5.1.7. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A such that $p_2(A) \neq R$. Let H be a strong subdigraph of Λ induced by $G \times p_2(A)$. If $Y = \{y \in G : y \in p_1(W)\} \times \{\gamma \in R : \gamma \notin p_2(A)\}$ where W is an $\alpha-$set of H, then $\alpha(D) \geq \max\{\alpha(H) + |Y|, (|R| - |p_2(A)|)|G|\}$.*

Proof. Suppose that A is a connection set of Λ in which $p_2(A) \neq R$ and let $U = \{(x, \xi) \in G \times R : \xi \notin p_2(A)\}$. Thus $U \neq \emptyset$. Let $(y, \beta), (z, \gamma) \in U$. If $((y, \beta), (z, \gamma)) \in E(\Lambda)$, that is, $(z, \gamma) = (y, \beta)(a, \lambda)$ for some $(a, \lambda) \in A$, then $\gamma = \beta\lambda = \lambda \in p_2(A)$ which is impossible. Similarly, if $((z, \gamma), (y, \beta)) \in E(\Lambda)$, then we also get a contradiction. Thus U is an independent set of Λ which implies that $\alpha(\Lambda) \geq |U| = (|R| - |p_2(A)|)|G|$.

Next, we will let H be a strong subdigraph of Λ which is induced by $G \times p_2(A)$ and $Y = \{y \in G : y \in p_1(W)\} \times \{\gamma \in R : \gamma \notin p_2(A)\}$ such that W is an $\alpha-$set of H. Thus $W \cap Y = \emptyset$. We will show that $W \cup Y$ is an independent set of Λ. Let $(x, \delta), (y, \eta) \in W \cup Y$. If $(x, \delta), (y, \eta) \in W$, then they are independent in H by the property of an independent set W. Hence they are also independent in Λ because H is the strong subdigraph of Λ. If $(x, \delta), (y, \eta) \in Y$ and $Y \subseteq U$ which is defined as above, then (x, δ) and (y, η) are independent since U is an independent set of Λ. Now, we consider in the case where one is in W and another one is in Y. Without loss of generality, we can assume that $(x, \delta) \in W$ and $(y, \eta) \in Y$. By the property of the Cayley digraph Λ of a right group $G \times R$, we have $((x, \delta), (y, \eta)) \notin E(\Lambda)$. If $((y, \eta), (x, \delta)) \in E(\Lambda)$, that is, $(x, \delta) = (y, \eta)(a, \lambda)$ for some $(a, \lambda) \in A$, then $x = ya$ and $\delta = \eta\lambda = \lambda$. Since $y \in p_1(W)$, there exists $\mu \in p_2(W)$ such that $(y, \mu) \in W$ and

$$(x, \delta) = (ya, \delta) = (y, \mu)(a, \delta) = (y, \mu)(a, \lambda) \text{ where } (a, \lambda) \in A,$$

that is, $((y, \mu), (x, \delta)) \in E(H)$ which contradicts to the independence of W. Hence (x, δ) and (y, η) are independent in Λ. Therefore, $W \cup Y$ is an independent set of Λ which implies that $\alpha(\Lambda) \geq |W \cup Y| = |W| + |Y| = \alpha(H) + |Y|$. So we can conclude that $\alpha(\Lambda) \geq \max\{\alpha(H) + |Y|, (|R| - |p_2(A)|)|G|\}$, as required. $\square$

Now, we consider the value of the independence number $\alpha(\Lambda)$ of a Cayley digraph Λ of a right group $G \times R$ such that the connection set A satisfies the condition that $|R| \geq 2|p_2(A)|$.

Theorem 5.1.8. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A in which $p_2(A) \neq R$. If $|R| \geq 2|p_2(A)|$, then $\alpha(\Lambda) = (|R| - |p_2(A)|)|G|$.*

Proof. Let A be a connection set of Λ such that $p_2(A) \neq R$. The proof of this theorem will be shown in the form of contraposition. Assume that $\alpha(\Lambda) \neq (|R| - |p_2(A)|)|G|$, that is, either $\alpha(\Lambda) < (|R| - |p_2(A)|)|G|$ or $\alpha(\Lambda) > (|R| - |p_2(A)|)|G|$. Since $p_2(A) \neq R$, we can conclude that $\alpha(\Lambda) \geq (|R| - |p_2(A)|)|G|$ as shown in the proof of Theorem 5.1.7. Hence we now have $\alpha(\Lambda) > (|R| - |p_2(A)|)|G|$. Let I be an independent set of Λ such that $|I| = \alpha(\Lambda) > (|R| - |p_2(A)|)|G|$ which implies that $I \cap (G \times p_2(A)) \neq \emptyset$. Let $T = \{(y, \beta) \in I : \beta \in p_2(A)\}$. Thus $T \neq \emptyset$. For each $(z, \lambda) \in T$, there exists $(a, \lambda) \in A$ such that $((za^{-1}, \gamma), (z, \lambda)) \in E(\Lambda)$ for all $\gamma \in R \setminus p_2(A)$. Since I is independent, we conclude that $(za^{-1}, \gamma) \notin I$ for all $\gamma \in R \setminus p_2(A)$. Hence

$$\begin{aligned}|I \setminus T| &\leq [(|R| - |p_2(A)|)|G|] - [|p_1(T)|(|R| - |p_2(A)|)] \\ &= (|R| - |p_2(A)|)(|G| - |p_1(T)|)\end{aligned}$$

which implies that

$$\begin{aligned}(|R| - |p_2(A)|)|G| < |I| &= |T| + |I \setminus T| \\ &\leq |T| + (|R| - |p_2(A)|)(|G| - |p_1(T)|).\end{aligned}$$

So we obtain that

$$\begin{aligned}|T| &> (|R| - |p_2(A)|)|G| - (|R| - |p_2(A)|)(|G| - |p_1(T)|) \\ &= (|R| - |p_2(A)|)|p_1(T)|.\end{aligned}$$

Since $(|R| - |p_2(A)|)|p_1(T)| < |T| \leq |p_1(T) \times p_2(T)| = |p_1(T)||p_2(T)|$, we obtain that $|R| - |p_2(A)| < |p_2(T)|$. By the definition of the set T, we can observe that $p_2(T) \subseteq p_2(A)$ which implies that $|R| - |p_2(A)| < |p_2(T)| \leq |p_2(A)|$. Consequently, we obtain that $|R| < |p_2(A)| + |p_2(A)| = 2|p_2(A)|$, this completes the proof. $\square$

Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. The digraph Λ is said to be *almost complete* if for each two different elements x and y in G satisfy the following condition:

$$((x, \beta), (y, \gamma)), ((y, \gamma), (x, \beta)) \in E(D) \text{ for all } \beta, \gamma \in p_2(A).$$

We now obtain the following proposition.

Proposition 5.1.9. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A such that $p_2(A) \neq R$. If the digraph* $\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)$ *is almost complete, then* $\alpha(\Lambda) = \max\left\{\frac{|G||R|}{|\langle p_1(A)\rangle|}, (|R| - |p_2(A)|)|G|\right\}$.

Proof. Assume that $\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)$ is almost complete. By the first part of the proof of Theorem 5.1.7, we have $\alpha(\Lambda) \geq (|R| - |p_2(A)|)|G|$. Since $p_1(A)$ does not contain the identity of G, we obtain that the set $\{x\} \times R$ is an independent set of Λ for all $x \in \langle p_1(A)\rangle$. Thus $\alpha(\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)) \geq |R|$ and then $\alpha(\Lambda) \geq \frac{|G||R|}{|\langle p_1(A)\rangle|}$. Hence $\alpha(\Lambda) \geq \max\left\{\frac{|G||R|}{|\langle p_1(A)\rangle|}, (|R| - |p_2(A)|)|G|\right\}$.

Now, let X be an $\alpha-$set of $\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)$. Suppose that there exist two different elements $x, y \in \langle p_1(A)\rangle$ in which $(x, \beta), (y, \gamma) \in X$ for some $\beta \in p_2(A)$ and $\gamma \in R$. Since the digraph $\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)$ is almost complete, we obtain that $((y, \beta), (x, \beta)) \in E(\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A))$, that is, $(x, \beta) = (y, \beta)(a, \lambda)$ for some $(a, \lambda) \in A$. Hence $x = ya$ and $\beta = \lambda$. We also have $(x, \beta) = (ya, \lambda) = (y, \gamma)(a, \lambda)$ which implies that $((y, \gamma), (x, \beta)) \in E(\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A))$. This contradicts to the property of the independent set X. Thus all elements in X must be either elements in $\{z\} \times R$, for some $z \in \langle p_1(A)\rangle$, or elements in $\langle p_1(A)\rangle \times (R \setminus p_2(A))$. So we can conclude that $\alpha(\Lambda) = \max\left\{\frac{|G||R|}{|\langle p_1(A)\rangle|}, (|R| - |p_2(A)|)|G|\right\}$, as required. $\square$

We next show the following theorem which gives an upper bound and a lower bound of the independence number of Cayley digraphs of right groups with respect to some specific connection sets.

Theorem 5.1.10. *Let Λ denote a Cayley digraph of a right group $G \times R$ with a connection set A in which $p_2(A) = R$. Then* $|R| \leq \alpha(\Lambda) \leq \left\lfloor\frac{|\langle p_1(A)\rangle|}{2}\right\rfloor \frac{|G||R|}{|\langle p_1(A)\rangle|}$.

Proof. Let A be a connection set of Λ such that $p_2(A) = R$. Since we consider the connection set A in the case where $e \notin p_1(A)$, we can observe that, for each $g \in G$, the set $\{g\} \times R$ is an independent set of Λ. Hence $\alpha(\Lambda) \geq |\{g\} \times R| = |R|$.

In order to verify the upper bound of $\alpha(\Lambda)$, we will assume to the contrary that $\alpha(D) > \left\lfloor\frac{|\langle p_1(A)\rangle|}{2}\right\rfloor \frac{|G||R|}{|\langle p_1(A)\rangle|}$. By applying Lemma 2.2.7, we can conclude that there exists $g \in G$ such that $\alpha((g\langle p_1(A)\rangle \times R), E_g) > \left\lfloor\frac{|\langle p_1(A)\rangle|}{2}\right\rfloor |R|$. Assume that X is an $\alpha-$set of the strong subdigraph $((g\langle p_1(A)\rangle \times R), E_g)$ which means that $|X| > \left\lfloor\frac{|\langle p_1(A)\rangle|}{2}\right\rfloor |R|$ and then $|X \cap (g\langle p_1(A)\rangle \times \{\lambda\})| > \left\lfloor\frac{|\langle p_1(A)\rangle|}{2}\right\rfloor$ for some $\lambda \in R$. Let $X_\lambda := X \cap (g\langle p_1(A)\rangle \times \{\lambda\})$ and $A_\lambda := A \cap (G \times \{\lambda\})$. It is not hard to investigate that $X_\lambda \cup X_\lambda A_\lambda \subseteq g\langle p_1(A)\rangle \times \{\lambda\}$ and $X_\lambda \cap X_\lambda A_\lambda = \emptyset$. We have $|X_\lambda \cup X_\lambda A_\lambda| \leq |g\langle p_1(A)\rangle \times \{\lambda\}| = |g\langle p_1(A)\rangle| = |\langle p_1(A)\rangle|$ and hence $|\langle p_1(A)\rangle| \geq |X_\lambda \cup X_\lambda A_\lambda| = |X_\lambda| + |X_\lambda A_\lambda| \geq 2|X_\lambda| \geq 2\left(\frac{|\langle p_1(A)\rangle|+k}{2}\right) > |\langle p_1(A)\rangle|$ for some $k \in \mathbb{N}$. This gives a contradiction. Thus the upper bound is proved. $\square$

The sharpness of those bounds given in the above theorem is described as follows.

Proposition 5.1.11. *The bounds stated in the above theorem are sharp.*

Proof. We first consider the lower bound of $\alpha(\Lambda)$. Let $A = (G \setminus \{e\}) \times R$ be a connection set of Λ where e is the identity of a group G. It is uncomplicated to examine that Λ is an almost complete digraph. It follows that $\{g\} \times R$ is an $\alpha-$set of Λ for all $g \in G$ that means $\alpha(\Lambda) = |\{g\} \times R| = |R|$.

For the sharpness of an upper bound of $\alpha(\Lambda)$, we shall consider the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_n \times R, A)$ of a right group $\mathbb{Z}_n \times R$ with a connection set $A = \{a\} \times R$ where n is a natural number greater than 1 and $a \neq e$. Suppose that $|a| = k$ for some $k \in \mathbb{N}$. If k is even, then the set $\{a, a^3, a^5, \ldots, a^{k-1}\} \times R$ is an $\alpha-$set of $\mathrm{Cay}(\langle p_1(A)\rangle \times R, A)$ and if k is odd, then the set $\{a, a^3, a^5, \ldots, a^{k-2}\} \times R$ is an $\alpha-$set of $\mathrm{Cay}(\langle p_1(A)\rangle \times R, A)$ which each of these two sets has the cardinality $\left\lfloor \frac{k}{2} \right\rfloor |R|$. Hence $\alpha(\mathrm{Cay}(\langle p_1(A)\rangle \times R, A)) = \left\lfloor \frac{k}{2} \right\rfloor |R| = \left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor |R|$ which certainly implies that $\alpha(\Lambda) = \left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor \frac{|G||R|}{|\langle p_1(A)\rangle|}$, as desired. $\square$

5.2 Weakly Independence Number

This section provides the results of a weakly independence number $\alpha_w(\Delta)$ of a Cayley digraph Δ of a rectangular group $G \times L \times R$ relative to a connection set A. The following two theorems are directly obtained by applying the similar arguments of Theorems 5.1.1 and 5.1.2, respectively.

Theorem 5.2.1. *If I is an α_w-set of Δ, then $I \cap (G \times \{\ell\} \times R)$ is an α_w-set of the digraph $(G \times \{\ell\} \times R, E_\ell)$ for all $\ell \in L$.*

Theorem 5.2.2. *If T is an α_w-set of* $\mathrm{Cay}(G \times R, \overline{A})$, *then* $\bigcup\limits_{(t,\lambda)\in T} (\{t\} \times L \times \{\lambda\})$ *is an α_w-set of Δ.*

Now, we give some facts about the properties of the weakly independence number $\alpha_w(\Gamma)$ of a Cayley digraph Γ of a left group $G \times L$ with respect to the connection set A.

Theorem 5.2.3. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. If $p_1(A) = [p_1(A)]^{-1}$ where $[p_1(A)]^{-1} = \{x^{-1} : x \in p_1(A)\}$, then $\alpha_w(\Gamma) = \alpha(\Gamma)$.*

Proof. Let A be a connection set of Γ in which $p_1(A) = [p_1(A)]^{-1}$. In fact, we have $\alpha(\Gamma) \leq \alpha_w(\Gamma)$, so we only need to prove that $\alpha_w(\Gamma) \leq \alpha(\Gamma)$. Let I be an α_w-set of Γ

and $(x,s),(y,t) \in I$. We claim that I is also an independent set of Γ. Assume to the contrary that those two elements (x,s) and (y,t) are not independent. We can assume without loss of generality that $((x,s),(y,t)) \in E(\Gamma)$. Thus $(y,t) = (x,s)(z,l) = (xz,s)$ for some $(z,l) \in A$, that is, $t = s$ and $y = xz$ where $z \in p_1(A) = [p_1(A)]^{-1}$. Then there exists $a \in p_1(A)$ such that $z = a^{-1}$ and there exists $w \in p_2(A)$ in which $(a,w) \in A$. We obtain that $(x,s) = (yz^{-1},t) = (ya,t) = (y,t)(a,w)$ and then $((y,t),(x,s)) \in E(\Gamma)$. Hence (x,s) and (y,t) are not weakly independent which contradicts to the property of I. Thus (x,s) and (y,t) are independent, this means that I is an independent set of Γ. We can conclude that $\alpha_w(\Gamma) = |I| \le \alpha(\Gamma)$, proving the assertion. □

Let A and $A^\sharp$ be connection sets of Cayley digraphs of a left group $G \times L$ such that $p_1(A^\sharp) = p_1(A) \cap [p_1(A)]^{-1}$. It is true that $p_1(A^\sharp) \subseteq \langle p_1(A) \rangle$, clearly. We consequently have the following theorem.

Theorem 5.2.4. $\alpha_w(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = \alpha_w(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A^\sharp)))$.

Proof. Let $C := \mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ and $C^\sharp := \mathrm{Cay}(\langle p_1(A)\rangle, p_1(A^\sharp))$ be Cayley digraphs of $\langle p_1(A)\rangle$ with connection sets $p_1(A)$ and $p_1(A^\sharp)$, respectively. Since $p_1(A^\sharp) \subseteq p_1(A)$ as defined above, we can conclude that $C^\sharp$ is a spanning subdigraph of C which implies that $\alpha_w(C) \le \alpha_w(C^\sharp)$, absolutely.

We now assume that X is an α_w-set of $C^\sharp$ and need to show that X is a weakly independent set in C. Suppose that there exist $x, y \in X$ such that $(x,y),(y,x) \in E(C)$. Then $y = xa$ and $x = yb$ for some $a, b \in p_1(A)$. We obtain that $y = xa = yba$ which implies that $ba = e$, the identity of a group G, that is, $b = a^{-1}$. Thus $a, a^{-1} \in p_1(A) \cap [p_1(A)]^{-1} = p_1(A^\sharp)$. This implies that $(x,y),(y,x) \in E(C^\sharp)$ which is a contradiction since x and y are weakly independent in $C^\sharp$. Hence $(x,y) \notin E(C)$ or $(y,x) \notin E(C)$, that is, x and y are weakly independent in C which leads to the fact that X is a weakly independent set of C. We can conclude that $\alpha_w(C^\sharp) = |X| \le \alpha_w(C)$. Therefore, $\alpha_w(C) = \alpha_w(C^\sharp)$ which certainly completes the proof as required. □

Let $\Gamma := \mathrm{Cay}(G \times L, A)$ and $\Gamma^\sharp := \mathrm{Cay}(G \times L, A^\sharp)$ be Cayley digraphs of a left group $G \times L$ with respect to connection sets A and $A^\sharp$, respectively, where $A^\sharp$ is the connection set as defined previously. We completely obtain by the definition of $A^\sharp$ that $p_1(A^\sharp) \subseteq p_1(A)$ which implies that $\alpha(D) \le \alpha(D^\sharp)$. Moreover, it is not hard to investigate that $p_1(A^\sharp) = [p_1(A^\sharp)]^{-1}$. We can conclude by Theorem 5.2.3 that $\alpha(\Gamma^\sharp) = \alpha_w(\Gamma^\sharp)$. By applying Theorem 5.2.4, we have $\alpha_w(\Gamma^\sharp) = \alpha_w(\Gamma)$. Hence we evidently get the following result about the weakly independence number of Cayley digraphs of left groups.

Theorem 5.2.5. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. Then* $\frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lceil \frac{|\langle p_1(A)\rangle|}{2|p_1(A)|+1}\right\rceil \leq \alpha_w(\Gamma) \leq \frac{|G||L|}{|\langle p_1(A^\sharp)\rangle|}\left\lfloor \frac{|\langle p_1(A^\sharp)\rangle|}{2}\right\rfloor$.

Next, we will consider the weakly independence number of a Cayley digraph Λ of a right group $G \times R$ with respect to a connection set A. We consequently have the following results.

Theorem 5.2.6. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. Then $\alpha_w(\Lambda) = |G||R|$ if and only if $p_1(A) \cap [p_1(A)]^{-1} = \emptyset$.*

Proof. Assume that $\alpha_w(\Lambda) = |G||R|$. Then $G \times R$ is a weakly independent set of Λ, this means that every two vertices of Λ are weakly independent. Suppose that there exists an element $a \in p_1(A) \cap [p_1(A)]^{-1}$. Thus $a = b^{-1}$ for some $b \in p_1(A)$. Hence there exist $\beta, \gamma \in p_2(A)$ such that $(a, \beta), (b, \gamma) \in A$. Let x be any element in G. Since $(xa, \beta) = (x, \gamma)(a, \beta)$ and $(x, \gamma) = (xaa^{-1}, \gamma) = (xa, \beta)(a^{-1}, \gamma) = (xa, \beta)(b, \gamma)$, we obtain that $((x, \gamma), (xa, \beta)), ((xa, \beta), (x, \gamma)) \in E(\Lambda)$, that is, (x, γ) and (xa, β) are not weakly independent in Λ which is a contradiction. Therefore, $p_1(A) \cap [p_1(A)]^{-1} = \emptyset$.

On the other hand, we will suppose that $p_1(A) \cap [p_1(A)]^{-1} = \emptyset$. If there exist two elements $(x, \beta), (y, \gamma) \in G \times R$ such that they are not weakly independent in Λ, that is, $((x, \beta), (y, \gamma)), ((y, \gamma), (x, \beta)) \in E(\Lambda)$, then $(y, \gamma) = (x, \beta)(a, \lambda) = (xa, \lambda)$ and $(x, \beta) = (y, \gamma)(b, \mu) = (yb, \mu)$ for some $(a, \lambda), (b, \mu) \in A$. Thus $y = xa$ and $x = yb$ and then $x = yb = xab$. Hence we have by the cancellation law that $a = b^{-1} \in [p_1(A)]^{-1}$. Consequently, $a \in p_1(A) \cap [p_1(A)]^{-1}$ which contradicts to our supposition. Therefore, $G \times R$ is a weakly independent set of Λ which completely implies that $\alpha_w(\Lambda) = |G||R|$, proving the assertion. □

For the connection set A of Λ in which $p_1(A) \cap [p_1(A)]^{-1} \neq \emptyset$, we obtain the lower bound of a weakly independence number $\alpha_w(\Lambda)$ of Λ as follows.

Theorem 5.2.7. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A in which $p_1(A) \cap [p_1(A)]^{-1} \neq \emptyset$. If $X = \{\lambda \in p_2(A) : a^{-1} \notin p_1(A)$ for all $(a, \lambda) \in A\}$, then $\alpha_w(\Lambda) \geq (|R| - |p_2(A)| + |X|)|G| + |p_2(A)| - |X|$.*

Proof. Let A be a connection set of Λ such that $p_1(A) \cap [p_1(A)]^{-1} \neq \emptyset$ and we define $X = \{\lambda \in p_2(A) : a^{-1} \notin p_1(A)$ for all $(a, \lambda) \in A\}$.

If $p_2(A) \neq R$, then $G \times (R \setminus p_2(A)) \neq \emptyset$ and it is not difficult to verify that the set $G \times (R \setminus p_2(A))$ is a weakly independent set of Λ.

We now show that $G \times X$ is a weakly independent set of Λ. Assume that there exist $(g_1, \lambda_1), (g_2, \lambda_2) \in G \times X$ such that $((g_1, \lambda_1), (g_2, \lambda_2)), ((g_2, \lambda_2), (g_1, \lambda_1)) \in E(\Lambda)$. Then $(g_2, \lambda_2) = (g_1, \lambda_1)(a, \beta) = (g_1 a, \beta)$ and $(g_1, \lambda_1) = (g_2, \lambda_2)(b, \gamma) = (g_2 b, \gamma)$ for some $(a, \beta), (b, \gamma) \in A$. Thus $g_2 = g_1 a, \lambda_2 = \beta$ and $g_1 = g_2 b, \lambda_1 = \gamma$. Hence $g_1 = g_2 b = g_1 ab$ which implies that $a^{-1} = b$. Hence there exists $(a, \beta) \in A$ such that $a^{-1} = b \in p_1(A)$, that is, $\lambda_2 = \beta \notin X$ which is a contradiction. So $G \times X$ is weakly independent in Λ.

For fixed $g \in G$, since the identity $e \notin p_1(A)$, we can conclude that $\{g\} \times (p_2(A) \setminus X)$ is weakly independent in Λ.

Next, we will prove that $I := [G \times (R \setminus p_2(A))] \cup (G \times X) \cup [\{g\} \times (p_2(A) \setminus X)]$ is a weakly independent set of Λ. Assume to the contrary that there exist $(c, \delta), (d, \eta) \in I$ such that $((c, \delta), (d, \eta)), ((d, \eta), (c, \delta)) \in E(\Lambda)$. We need to consider the following two cases.

Case (i): if $(c, \delta) \in G \times (R \setminus p_2(A))$ and (d, η) is in $G \times X$ or $\{g\} \times (p_2(A) \setminus X)$, then $\delta \notin p_2(A)$. Since $((d, \eta), (c, \delta)) \in E(\Lambda)$, we get that there exists $(k, \varepsilon) \in A$ such that $(c, \delta) = (d, \eta)(k, \varepsilon)$ and then $\delta = \eta\varepsilon = \varepsilon \in p_2(A)$, a contradiction.

Case (ii): if $(c, \delta) \in G \times X$ and $(d, \eta) \in \{g\} \times (p_2(A) \setminus X)$, then $\delta \in X$ that means $a^{-1} \notin p_1(A)$ for all $(a, \delta) \in A$. Since $((c, \delta), (d, \eta)), ((d, \eta), (c, \delta)) \in E(\Lambda)$, there exist $(u, \zeta), (v, \xi) \in A$ such that $(d, \eta) = (c, \delta)(u, \zeta) = (cu, \zeta)$ and $(c, \delta) = (d, \eta)(v, \xi) = (dv, \xi)$. Then $d = cu, \eta = \zeta$ and $c = dv, \delta = \xi$. Thus $d = cu = dvu$ which implies that $u = v^{-1}$. Hence $v^{-1} = u \in p_1(A)$ where $(v, \xi) \in A$ and whence $\delta = \xi \notin X$, this gives a contradiction. Therefore, the set I is a weakly independent set of Λ, that is, $\alpha_w(\Lambda) \geq |I|$. Since $G \times (R \setminus p_2(A))$, $G \times X$ and $\{g\} \times (p_2(A) \setminus X)$ are pairwise disjoint, we obtain that

$$\begin{aligned}
\alpha_w(D) &\geq |[G \times (R \setminus p_2(A))] \cup (G \times X) \cup [\{g\} \times (p_2(A) \setminus X)]| \\
&= |G \times (R \setminus p_2(A))| + |G \times X| + |\{g\} \times (p_2(A) \setminus X)| \\
&= |G|(|R| - |p_2(A)|) + |G||X| + |p_2(A)| - |X| \\
&= |G|(|R| - |p_2(A)| + |X|) + |p_2(A)| - |X|.
\end{aligned}$$

This completes the proof of the theorem. □

Next, we will present the exact value of a weakly independence number of Cayley digraphs of right groups with respect to some special connection sets. In order to state the theorem, we need to show the following lemma which is useful for proving our theorem.

Lemma 5.2.8. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A in which $p_1(A) \cap [p_1(A)]^{-1} \neq \emptyset$. Let e be the identity of a group G and define $X = \{(x, \lambda) \in A : x \in p_1(A) \cap [p_1(A)]^{-1}$ and $x^2 \neq e\}$ where $|p_1(X)| = |X| = |p_2(X)|$. If W is a weakly independent set of Λ, then W contains at most $\frac{1}{2}|G \times p_2(X)|$ elements of $G \times p_2(X)$.*

Proof. Suppose that the conditions hold. Let W be a weakly independent set of Λ. Assume, contrary to what we want to prove, that $|W \cap (G \times p_2(X))| > \frac{1}{2}|G \times p_2(X)|$. Let $\{J, K\}$ be a partition of $p_1(X)$ satisfying the condition that for each $z \in J$, z^{-1} must belong to K. Thus $|J| = |K| = \frac{1}{2}|G \times p_2(X)|$. Since $|p_1(X)| = |X| = |p_2(X)|$, we can conclude that for each $w \in p_1(X)$, there exists a unique $\beta \in p_2(X)$ such that $(w, \beta) \in X$ and for each $\gamma \in p_2(X)$, there exists a unique $u \in p_1(X)$ such that $(u, \gamma) \in X$. Let

$$I = \{\zeta \in p_2(X) : (z, \zeta) \in X \text{ for some } z \in J\} \text{ and}$$
$$L = \{\xi \in p_2(X) : (k, \xi) \in X \text{ for some } k \in K\}.$$

We can observe that $\{I, L\}$ forms a partition of $p_2(X)$ with $|I| = |J| = |L|$.

Next, let (g, λ) be any element in $G \times p_2(X)$. We will show that there exists a unique $(x, \mu) \in G \times p_2(X)$ such that (g, λ) is not weakly independent to (x, μ). Suppose that there exist $(x_1, \mu_1), (x_2, \mu_2) \in G \times p_2(X)$ such that they are not weakly independent to (g, λ). Without loss of generality, we assume that $(g, \lambda) \in G \times I$. Hence there exist $a_1, a_2 \in p_1(A)$ such that $g = x_1 a_1$ where $(a_1, \lambda), (a_1^{-1}, \mu_1) \in A$ and $g = x_2 a_2$ where $(a_2, \lambda), (a_2^{-1}, \mu_2) \in A$, respectively. Since $(a_1, \lambda), (a_2, \lambda) \in A$ and $a_1, a_2 \in p_1(X)$, we can conclude that $a_1 = a_2$ which implies that $\mu_1 = \mu_2$. Then $x_1 = g a_1^{-1} = g a_2^{-1} = x_2$. Hence $(x_1, \mu_1) = (x_2, \mu_2)$. Since $\{I, L\}$ is a partition of $p_2(X)$, we have $\{G \times I, G \times L\}$ is a partition of $G \times p_2(X)$. For any weakly independent set U, we define U^* to be the set consisting of all vertices that are not weakly independent to any vertex in U, that is, $U^* = \{v : v \text{ is not weakly independent to } u \text{ for some } u \in U\}$. Consider

$$W \cap (G \times p_2(X)) = W \cap [(G \times I) \cup (G \times L)] = [W \cap (G \times I)] \cup [W \cap (G \times L)] \text{ and}$$
$$[W \cap (G \times I)] \cup [W \cap (G \times L)] \cup [W \cap (G \times I)]^* \cup [W \cap (G \times L)]^* \subseteq G \times p_2(X),$$

then
$$\begin{aligned} |G \times p_2(X)| &\geq |[W \cap (G \times I)] \cup [W \cap (G \times L)]| + |[W \cap (G \times I)]^* \cup [W \cap (G \times L)]^*| \\ &> \tfrac{1}{2}|G \times p_2(X)| + \tfrac{1}{2}|G \times p_2(X)| \\ &= |G \times p_2(X)| \text{ which is a contradiction.} \end{aligned}$$

This completes the proof of our assertion, as required. □

Now, we are ready to present the following theorem.

Theorem 5.2.9. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A in which $p_1(A) \cap [p_1(A)]^{-1} \neq \emptyset$. If $Y = \{(y, \lambda) \in A : y \in p_1(A) \cap [p_1(A)]^{-1}\}$ and $|p_1(Y)| = |Y| = |p_2(Y)|$, then $\alpha_w(\Lambda) = |G|\left(|R| - \frac{|Y|}{2}\right)$.*

Proof. Let $Y = \{(y, \lambda) \in A : y \in p_1(A) \cap [p_1(A)]^{-1}\}$ be such that $|p_1(Y)| = |Y| = |p_2(Y)|$. We first show that $G \times (R \setminus p_2(Y))$ is a weakly independent set of Λ. Suppose that there exist $(g_1,\lambda_1),(g_2,\lambda_2)\in G\times(R\setminus p_2(Y))$ such that $((g_1,\lambda_1),(g_2,\lambda_2)),((g_2,\lambda_2),(g_1,\lambda_1))\in E(\Lambda)$. Then $(g_2,\lambda_2)=(g_1,\lambda_1)(a_1,\mu_1)=(g_1a_1,\mu_1)$ and $(g_1,\lambda_1)=(g_2,\lambda_2)(a_2,\mu_2)=(g_2a_2,\mu_2)$ for some $(a_1,\mu_1),(a_2,\mu_2)\in A$. Thus $g_2=g_1a_1,\lambda_2=\mu_1$ and $g_1=g_2a_2,\lambda_1=\mu_2$. We can obtain that $g_2=g_1a_1=g_2a_2a_1$ which implies that $a_2=a_1^{-1}$. Hence $a_1,a_2\in p_1(A)\cap[p_1(A)]^{-1}$ and then $(a_1,\mu_1),(a_2,\mu_2)\in Y$, that is, μ_1 and μ_2 must belong to $p_2(Y)$. We can conclude that $\lambda_1,\lambda_2\in p_2(Y)$ which is a contradiction. Therefore, $G\times(R\setminus p_2(Y))$ is a weakly independent set of Λ and it follows that $\alpha_w(\Lambda)\geq|G|(|R|-|p_2(Y)|)$.

Now, let $T=\{(a,\lambda)\in Y:a=a^{-1}\}$ and $Z=Y\setminus T$. By the definition of the set Z, we can conclude that for each $z\in p_1(Z)$, z^{-1} must belong to $p_1(Z)$. Let $\{J,K\}$ be a partition of $p_1(Z)$ which satisfies that for each $x\in J$, x^{-1} must belong to K. Thus $|J|=|K|=\frac{|p_1(Z)|}{2}$. Since $Z\subseteq Y$ and $|p_1(Y)|=|Y|=|p_2(Y)|$, we have $|p_1(Z)|=|Z|=|p_2(Z)|$. Then for each $w\in p_1(Z)$, there exists a unique $\beta\in p_2(Z)$ such that $(w,\beta)\in Z$ and for each $\gamma\in p_2(Z)$, there exists a unique $u\in p_1(Z)$ such that $(u,\gamma)\in Z$. Let $I=\{\eta\in p_2(Z):(z,\eta)\in Z \text{ for some } z\in J\}$. We will prove that $G\times I$ is a weakly independent set of Λ. Let $(g_1,\lambda_1),(g_2,\lambda_2)\in G\times I$. Since $\lambda_1,\lambda_2\in I$, we obtain that $(z_1,\lambda_1),(z_2,\lambda_2)\in Z$ for some $z_1,z_2\in J$. If $((g_1,\lambda_1),(g_2,\lambda_2)),((g_2,\lambda_2),(g_1,\lambda_1))\in E(\Lambda)$, then there exist $(a,\zeta),(b,\xi)\in A$ such that $(g_2,\lambda_2)=(g_1,\lambda_1)(a,\zeta)=(g_1a,\zeta)$ and $(g_1,\lambda_1)=(g_2,\lambda_2)(b,\xi)=(g_2b,\xi)$. Thus $g_2=g_1a,\lambda_2=\zeta$ and $g_1=g_2b,\lambda_1=\xi$ which implies that $b=a^{-1}$. We have $a,b\in p_1(A)\cap[p_1(A)]^{-1}$ which follows that $(a,\zeta),(b,\xi)\in Y$. Since $Z\subseteq Y$ and $|p_1(Y)|=|Y|=|p_2(Y)|$, we have $a=z_2$ and $b=z_1$ and hence $a,b\in J$ which contradicts to the definition of J. Thus $G\times I$ is weakly independent in Λ that means $\alpha_w(\Lambda)\geq|G\times I|=|G||I|=|G||J|=|G|\frac{|p_1(Z)|}{2}=\frac{1}{2}|G|(|p_1(Y)|-|p_1(T)|)$.

For each $(a,\lambda)\in T$, we obtain that $\mathrm{Cay}(G\times\{\lambda\},\{(a,\lambda)\})$ is the disjoint union of $\frac{|G|}{2}$ strong subdigraphs which each subdigraph is isomorphic to a strongly connected digraph of order 2. By choosing one vertex from each subdigraph, we can conclude that the set of these chosen vertices forms a weakly independent set of $\mathrm{Cay}(G\times\{\lambda\},\{(a,\lambda)\})$. It is not difficult to verify that $\alpha_w(\mathrm{Cay}(G\times\{\lambda\},\{(a,\lambda)\}))=\frac{|G|}{2}$. Since every vertex in $G\times\{\lambda\}$ is weakly independent to other vertices in $G\times(R\setminus\{\lambda\})$, we obtain that

$$\alpha_w(\mathrm{Cay}(G\times p_2(T),T))=\frac{|G|}{2}|T|.$$

Moreover, we can conclude that every vertex in $G\times p_2(Z)$ is weakly independent to every vertex in $G\times p_2(T)$. Therefore,

$$\begin{aligned}\alpha_w(\Lambda)&\geq\frac{|G|}{2}(|p_1(Y)|-|p_1(T)|)+\frac{|G|}{2}|T|+|G|(|R|-|p_2(Y)|)\\&=|G|\left[\frac{|p_1(Y)|}{2}-\frac{|p_1(T)|}{2}+\frac{|T|}{2}+|R|-|p_2(Y)|\right]\\&=|G|\left(|R|-\frac{|Y|}{2}\right).\end{aligned}$$

We now suppose that there exists a weakly independent set U of a digraph Λ such that $|U| > |G|\left(|R| - \frac{|Y|}{2}\right)$. It is easy to obtain that $G \times (R \setminus p_2(Y)) \subseteq U$. Since every vertex in $G \times p_2(T)$ is weakly independent to other vertices in $G \times (R \setminus p_2(T))$ and from $\alpha_w(\mathrm{Cay}(G \times p_2(T), T)) = \frac{|G|}{2}|T|$, we can conclude that U contains an α_w-set of $\mathrm{Cay}(G \times p_2(T), T)$. It follows that

$$\begin{aligned}
|U \cap (G \times p_2(Z))| &= |U| - |G|(|R| - |p_2(Y)|) - \tfrac{|G|}{2}|T| \\
&> |G|(|R| - \tfrac{|Y|}{2}) - |G|(|R| - |p_2(Y)|) - \tfrac{|G|}{2}|T| \\
&= |G|\left[|R| - \tfrac{|Y|}{2} - |R| + |p_2(Y)| - \tfrac{|T|}{2}\right] \\
&= |G|\left(\tfrac{|p_2(Y)|}{2} - \tfrac{|p_2(T)|}{2}\right) \\
&= |G|\tfrac{|p_2(Y \setminus T)|}{2} \\
&= |G|\tfrac{|p_2(Z)|}{2} \\
&= \tfrac{1}{2}|G \times p_2(Z)|
\end{aligned}$$

which contradicts to Lemma 5.2.8. Therefore, $\alpha_w(\Lambda) = |G|\left(|R| - \frac{|Y|}{2}\right)$. □

5.3 Dipath Independence Number

The purpose of this section is to investigate the dipath independence number of Cayley digraphs Δ, Γ, and Λ of a rectangular group $G \times L \times R$, a left group $G \times L$, and a right group $G \times R$, respectively.

We will start this section with the results of the dipath independence number of Cayley digraphs of rectangular groups with respect to their connection sets. By applying the similar arguments of Theorems 5.1.1 and 5.1.2, we respectively obtain the following results.

Theorem 5.3.1. *If I is an α_p-set of Δ, then $I \cap (G \times \{\ell\} \times R)$ is an α_p-set of the digraph $(G \times \{\ell\} \times R, E_\ell)$ for all $\ell \in L$.*

Theorem 5.3.2. *If T is an α_p-set of $\mathrm{Cay}(G \times R, \overline{A})$, then $\bigcup_{(t,\lambda) \in T} (\{t\} \times L \times \{\lambda\})$ is an α_p-set of Δ.*

Next, we shall consider the dipath independence number $\alpha_p(\Gamma)$ of a Cayley digraph Γ of a left group $G \times L$ with a connection set A. Since Γ is the disjoint union of strong subdigraphs which each subdigraph is isomorphic to $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$, we need to describe the useful property of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ in the following lemma which will be referred in the next theorem.

Lemma 5.3.3. *If A is a connection set of a digraph Γ, then* $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ *is strongly connected.*

Proof. Let A be a connection set of Γ and $x, y \in \langle p_1(A)\rangle$. We claim that there exists a dipath connecting between x and y. Since $\langle p_1(A)\rangle$ is a subgroup of G, there exists $a \in \langle p_1(A)\rangle$ such that $y = xa$. Then $a = a_1a_2\cdots a_k$ for some $a_1, a_2, \ldots, a_k \in p_1(A)$ and $k \in \mathbb{N}$. Thus $(x, xa_1), (xa_1, xa_1a_2), \ldots, (xa_1a_2\cdots a_{k-1}, y) \in E(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$ that means $x, xa_1, xa_1a_2, \ldots, xa_1a_2\cdots a_{k-1}, xa_1a_2\cdots a_{k-1}a_k = xa = y$ is the dipath connecting from x to y in $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$. So we can conclude that $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ is a strongly connected digraph as claimed. $\square$

The following theorem shows the exact value of a dipath independence number on Cayley digraphs of left groups with respect to their connection sets.

Theorem 5.3.4. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. Then $\alpha_p(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}$.*

Proof. By considering $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$, we consequently obtain by Lemma 5.3.3 that it is a strongly connected subdigraph of $\mathrm{Cay}(G, p_1(A))$. Hence there is a dipath that connects between any two different vertices of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ which implies that $\alpha_p(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = 1$. Furthermore, we can obtain by using Lemma 2.2.6 that $\alpha_p(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|} \cdot \alpha_p(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = \frac{|G||L|}{|\langle p_1(A)\rangle|}$. $\square$

The following theorem describes the relation between a weakly independence number and a dipath independence number of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ which can be isomorphically considered as a strong subdigraph of Γ.

Theorem 5.3.5. *Let A be a connection set of Γ and denote by C the Cayley digraph* $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ *of $\langle p_1(A)\rangle$ with a connection set $p_1(A)$. Then the following statements are equivalent:*

(1). $\alpha_p(C) = \alpha_w(C)$;

(2). $\alpha_w(C) = 1$;

(3). $p_1(A) \cup \{e\} = \langle p_1(A)\rangle$ *where e is the identity of a group G.*

Proof. Let $C := \mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ be a Cayley digraph of a group $\langle p_1(A)\rangle$ with a connection set $p_1(A)$.

(1) $\Rightarrow$ (2) : Suppose that $\alpha_p(C) = \alpha_w(C)$. As we have known from Lemma 5.3.3 that C is strongly connected, we get that $\alpha_p(C) = 1$ and whence $\alpha_w(C) = 1$.

(2) $\Rightarrow$ (3) : Assume that $\alpha_w(C) = 1$. It is clear that $p_1(A) \cup \{e\} \subseteq \langle p_1(A) \rangle$. We now let $x \in \langle p_1(A) \rangle$. If $x = e$, then $x \in p_1(A) \cup \{e\}$, obviously. So we next consider in the case where $x \neq e$. Since $\alpha_w(C) = 1$, we obtain that $\{y\}$ is an α_w−set of C where $y \in \langle p_1(A) \rangle$ and then $(y, yx) \in E(C)$, that is, $yx = ya$ for some $a \in p_1(A)$. Thus $x = a \in p_1(A)$ by the left cancellation law. Hence the statement (3) is proved.

(3) $\Rightarrow$ (1) : Suppose that $p_1(A) \cup \{e\} = \langle p_1(A) \rangle$. For any two different vertices $x, y \in \langle p_1(A) \rangle$, we obtain that $(x, y), (y, x) \in E(C)$ since $y = x(x^{-1}y)$ and $x = y(y^{-1}x)$ where $x^{-1}y, y^{-1}x \in \langle p_1(A) \rangle \setminus \{e\} = p_1(A)$, respectively. Thus $\{x\}$ is an α_w−set of C and also a dipath independent set of C. Hence $\alpha_w(C) = |\{x\}| \leq \alpha_p(C)$. As we have generally known that $\alpha_p(C) \leq \alpha_w(C)$, we can absolutely conclude that $\alpha_p(C) = \alpha_w(C)$, this completes the proof. □

In general, if Γ is a Cayley digraph of a left group with any connection set A, we have the fact that $\alpha_p(\Gamma) \leq \alpha(\Gamma) \leq \alpha_w(\Gamma)$. Thus we directly obtain the following result.

Corollary 5.3.6. *Let A be a connection set of Γ and denote by C the Cayley digraph* $\mathrm{Cay}(\langle p_1(A) \rangle, p_1(A))$ *of $\langle p_1(A) \rangle$ with a connection set $p_1(A)$. Then $\alpha_p(C) = \alpha(C) = \alpha_w(C) = 1$ if and only if $p_1(A) \cup \{e\} = \langle p_1(A) \rangle$.*

Now, we illustrate the exact value of a dipath independence number of a Cayley digraph Λ of a right group $G \times R$ with respect to the connection set A.

Theorem 5.3.7. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. If $p_2(A) \neq R$, then $\alpha_p(\Lambda) = |G|(|R| - |p_2(A)|)$.*

Proof. Let A be a connection set of Λ such that $p_2(A) \neq R$. We will show that the set $G \times (R \setminus p_2(A))$ is a dipath independent set of Λ. Suppose that there exist two elements $(g_1, \lambda_1), (g_2, \lambda_2) \in G \times (R \setminus p_2(A))$ such that they are not dipath independent, that is, there exists a dipath from (g_1, λ_1) to (g_2, λ_2) in Λ or there exists a dipath from (g_2, λ_2) to (g_1, λ_1) in Λ. Without loss of generality, we can assume that Λ contains a dipath from (g_1, λ_1) to (g_2, λ_2). Hence we can write $(g_2, \lambda_2) = (g_1, \lambda_1)(a_1, \mu_1)(a_2, \mu_2) \cdots (a_k, \mu_k)$ for some $k \in \mathbb{N}$ where $(a_i, \mu_i) \in A$ for all $i \in \{1, 2, \ldots, k\}$. It follows that $g_2 = g_1 a_1 a_2 \cdots a_k$ and $\lambda_2 = \lambda_1 \mu_1 \mu_2 \cdots \mu_k = \mu_k \in p_2(A)$ which leads to a contradiction because $(g_2, \lambda_2) \in G \times (R \setminus p_2(A))$. Thus $G \times (R \setminus p_2(A))$ is a dipath independent set of Λ which directly implies that $\alpha_p(\Lambda) \geq |G \times (R \setminus p_2(A))| = |G|(|R| - |p_2(A)|)$.

Assume that there exists an α_p−set I such that $|I| > |G|(|R| - |p_2(A)|)$. Then there exists $x \in G$ such that $(x, \eta), (x, \rho) \in I$ for some $\eta \in p_2(A)$ and $\rho \in R \setminus p_2(A)$. Since $\eta \in p_2(A)$, there exists $a \in p_1(A)$ such that $(a, \eta) \in A$. Suppose that $|a| = k$ for some $k \in \mathbb{N}$. Hence $(x, \eta) = (xa^k, \eta) = (x, \rho)(a, \eta)^k$ which implies that there exists a dipath from (x, ρ) to (x, η), a contradiction. Consequently, $\alpha_p(\Lambda) = |G|(|R| - |p_2(A)|)$, as required. $\square$

Theorem 5.3.8. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. If $p_2(A) = R$, then $\alpha_p(\Lambda) = \frac{|G|}{|\langle p_1(A)\rangle|}$.*

Proof. Let A be a connection set of a digraph Λ such that $p_2(A) = R$. Assume that $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$ is the set of all left cosets of $\langle p_1(A)\rangle$ in G where $k \in \mathbb{N}$. By Lemma 2.2.7, we obtain that Λ is the disjoint union of k isomorphic strong subdigraphs $((g_i\langle p_1(A)\rangle \times R), E_i)$ of Λ such that $((g_i\langle p_1(A)\rangle \times R), E_i) \cong \mathrm{Cay}(\langle A\rangle, A)$ for all $i \in \{1, 2, \ldots, k\}$. So we need to consider a digraph $\mathrm{Cay}(\langle A\rangle, A)$ instead of Λ. We now show that $\mathrm{Cay}(\langle A\rangle, A)$ is strongly connected. Let $(x, \lambda), (y, \mu) \in \langle A\rangle$. By Lemma 2.2.8, we obtain that $\langle A\rangle = \langle p_1(A)\rangle \times p_2(A) = \langle p_1(A)\rangle \times R$. Since $\mu \in R$, there exists $a \in p_1(A)$ such that $(a, \mu) \in A$. From $x, y \in \langle p_1(A)\rangle$ and $\langle p_1(A)\rangle$ is a group, we can write $y = xu$ for some $u \in \langle p_1(A)\rangle$. Thus $u = u_1u_2\cdots u_t$ where $u_1, u_2, \ldots, u_t \in p_1(A)$. Then there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t \in R$ in which $(u_1, \varepsilon_1), (u_2, \varepsilon_2), \ldots, (u_t, \varepsilon_t) \in A$. Consider $(y, \varepsilon_t) = (xu, \varepsilon_t) = (xu_1u_2\cdots u_t, \varepsilon_t) = (x, \lambda)(u_1, \varepsilon_1)(u_2, \varepsilon_2)\cdots(u_t, \varepsilon_t)$, we can conclude that there exists a dipath from (x, λ) through to (y, ε_t). Assume that $|a| = n$ for some $n \in \mathbb{N}$. Therefore, $y = ya^n$ and then $(y, \mu) = (ya^n, \mu) = (y, \varepsilon_t)(a, \mu)^n$ which implies that there exists a dipath connecting from (y, ε_t) to (y, μ). So we can conclude that there exists a dipath from (x, λ) to (y, μ) which leads to the result that $\mathrm{Cay}(\langle A\rangle, A)$ is strongly connected. It is simple to investigate that $\alpha_p(\mathrm{Cay}(\langle A\rangle, A)) = 1$. For this reason, we can completely imply that $\alpha_p(\Lambda) = |G/\langle p_1(A)\rangle| \cdot \alpha_p(\mathrm{Cay}(\langle A\rangle, A)) = \frac{|G|}{|\langle p_1(A)\rangle|}$, as desired. $\square$

5.4 Weakly Dipath Independence Number

The aim of this section is to study the weakly dipath independence number of Cayley digraphs of rectangular groups with their connection sets. Especially, the weakly dipath independence number of Cayley digraphs of left groups and right groups are also considered.

We firstly describe the weakly dipath independence number $\alpha_{wp}(\Delta)$ of a Cayley digraph Δ of a rectangular group $G \times L \times R$ with a connection set A. The following theorems are directly obtained by applying the similar arguments of the results from Theorems 5.1.1 and 5.1.2.

Theorem 5.4.1. *If I is an $\alpha_{wp}-$set of Δ, then $I \cap (G \times \{\ell\} \times R)$ is an $\alpha_{wp}-$set of the digraph $(G \times \{\ell\} \times R, E_\ell)$ for all $\ell \in L$.*

Theorem 5.4.2. *If T is an $\alpha_{wp}-$set of $\mathrm{Cay}(G \times R, \overline{A})$, then $\bigcup_{(t,\lambda)\in T} (\{t\} \times L \times \{\lambda\})$ is an $\alpha_{wp}-$set of Δ.*

Now, we consider the weakly dipath independence number of Cayley digraphs of left groups with their connection sets. Actually, we can say that a Cayley digraph Γ of a left group $G \times L$ is the disjoint union of isomorphic subdigraphs which each of them is strongly connected by Lemma 5.3.3. This means that if there exists a dipath connecting from a vertex u to another vertex v in that Cayley digraph, there must be a dipath joining from v to u, as well. Consequently, the results of a weakly dipath independence number of Cayley digraphs of left groups can be considered to be the same results as a dipath independence number of Cayley digraphs of left groups, straightforwardly.

The last theorem in this section shows the weakly dipath independence number $\alpha_{wp}(\Lambda)$ of a Cayley digraph Λ of a right group $G \times R$ with an arbitrary connection set A.

Theorem 5.4.3. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. Then $\alpha_{wp}(\Lambda) = \frac{|G|}{|\langle p_1(A)\rangle|} + |G|(|R| - |p_2(A)|)$.*

Proof. Let A be a connection set of Λ. We now consider the following two cases.

Case (i): $p_2(A) = R$. Then $|G|(|R| - |p_2(A)|) = 0$. We have already known by Lemma 2.2.7 and Lemma 2.2.8 that the digraph Λ can be considered as the disjoint union of $\frac{|G|}{|\langle p_1(A)\rangle|}$ isomorphic strong subdigraphs $\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)$. In addition, we have already proved in Theorem 5.3.8 that $\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)$ is a strongly connected subdigraph of Λ. It can be easily shown that every two elements of $\langle p_1(A)\rangle \times p_2(A)$ are weakly dipath independent which directly implies that $\alpha_{wp}(\mathrm{Cay}(\langle p_1(A)\rangle \times p_2(A), A)) = 1$. So we can conclude that

$$\alpha_{wp}(\Lambda) = \frac{|G|}{|\langle p_1(A)\rangle|} + |G|(|R| - |p_2(A)|).$$

Case (ii): $p_2(A) \neq R$. We have proved in Theorem 5.3.7 that $G \times (R \setminus p_2(A))$ is a dipath independent set of Λ. It follows that $G \times (R \setminus p_2(A))$ is also a weakly dipath independent set of Λ. Consider $\mathrm{Cay}(G \times p_2(A), A)$, we can certainly obtain that $\alpha_{wp}(\mathrm{Cay}(G \times p_2(A), A)) = \frac{|G|}{|\langle p_1(A)\rangle|}$ by applying the proof of the above case. Since there is no dipath from any vertex in $G \times p_2(A)$ to any vertex in $G \times (R \setminus p_2(A))$, we can totally conclude that $\alpha_{wp}(\Lambda) \geq \frac{|G|}{|\langle p_1(A)\rangle|} + |G|(|R| - |p_2(A)|)$. It is not complicated to observe that every $\alpha_{wp}-$set of Λ must contain the set $G \times (R \setminus p_2(A))$. Hence if we let X be an $\alpha_{wp}-$set of Λ such that $|X| > \frac{|G|}{|\langle p_1(A)\rangle|} + |G|(|R| - |p_2(A)|)$, then there exist at least $\frac{|G|}{|\langle p_1(A)\rangle|} + 1$ elements in $G \times p_2(A)$ such that they are contained in X. Thus there exist at least two elements of X belonging to the same strongly connected subdigraph $\mathrm{Cay}(g\langle p_1(A)\rangle \times p_2(A), A)$ of Λ for some $g \in G$. Hence those elements are not weakly dipath independent in Λ which contradicts to the weakly dipath independence of X. Therefore,

$$\alpha_{wp}(\Lambda) = \frac{|G|}{|\langle p_1(A)\rangle|} + |G|(|R| - |p_2(A)|). \qquad \square$$

However, we can evidently observe from the results described in Theorem 5.3.8 and Theorem 5.4.3 that $\alpha_p(\Lambda) = \frac{|G|}{|\langle p_1(A)\rangle|} = \alpha_{wp}(\Lambda)$ in the case where $p_2(A) = R$.