CHAPTER 6

Independent Domination on Cayley Digraphs of Rectangular Groups

This chapter provides some results of independent domination parameters on Cayley digraphs of rectangular groups, left groups, and right groups with their connection sets. We divide this chapter into four parts comprising of the independent domination number, weakly independent domination number, dipath independent domination number, and weakly dipath independent domination number of those Cayley digraphs.

6.1 Independent Domination Number

In this section, we present some results about the independent domination number of Cayley digraphs of rectangular groups with their connection sets. Furthermore, some results on Cayley digraphs of left groups and right groups are also given.

Recall that we let Δ be a Cayley digraph $\mathrm{Cay}(G \times L \times R, A)$ of a rectangular group $G \times L \times R$ with a connection set A. The following theorem is directly obtained by applying Theorem 3.1.1.

Theorem 6.1.1. *If* $\overline{A} = \{(a,\alpha) \in G \times R : (a,l,\alpha) \in A$ *for some* $l \in L\}$*, then* $i(\Delta) = |L|(i(\mathrm{Cay}(G \times R, \overline{A})))$.

For each $l \in L$, we now clarify the corresponding relation between an $i-$set of Δ and an $i-$set of the strong subdigraph $(G \times \{l\} \times R, E_l)$ which is isomorphic to $\mathrm{Cay}(G \times R, \overline{A})$ as follows.

Theorem 6.1.2. *If* I *is an* $i-$*set of* Δ*, then* $I \cap (G \times \{l\} \times R)$ *is an* $i-$*set of the digraph* $(G \times \{l\} \times R, E_l)$ *for all* $l \in L$.

Proof. Let I be an $i-$set of Δ and $l \in L$. We will show that $I \cap (G \times \{l\} \times R)$ is an $i-$set of the digraph $(G \times \{l\} \times R, E_l)$. It is easy to see that $I \cap (G \times \{l\} \times R)$ is a nonempty subset of $G \times \{l\} \times R$. Since $I \cap (G \times \{l\} \times R) \subseteq I$, we get that $I \cap (G \times \{l\} \times R)$ is an independent set of $(G \times \{l\} \times R, E_l)$. Let $(g,l,\gamma) \in (G \times \{l\} \times R) \setminus [I \cap (G \times \{l\} \times R)]$. Then $(g,l,\gamma) \in (G \times \{l\} \times R) \setminus I$. Since I is a dominating set of Δ, there exists $(x,k,\delta) \in I$ such that $((x,k,\delta),(g,l,\gamma)) \in E(\Delta)$, that is, $(g,l,\gamma) = (x,k,\delta)(a,p,\tau) = (xa,k,\tau)$ for some $(a,p,\tau) \in A$ which implies that $l = k$. Thus $(x,k,\delta) = (x,l,\delta) \in G \times \{l\} \times R$ and hence $(x,k,\delta) \in I \cap (G \times \{l\} \times R)$. We can conclude that $((x,k,\delta),(g,l,\gamma)) \in E(G \times \{l\} \times R, E_l)$ that means $I \cap (G \times \{l\} \times R)$ is a dominating set of $(G \times \{l\} \times R, E_l)$. We obtain that $i(G \times \{l\} \times R, E_l) \le |I \cap (G \times \{l\} \times R)|$. Assume that there exists an independent dominating set X of $(G \times \{l\} \times R, E_l)$ in which $|X| < |I \cap (G \times \{l\} \times R)|$. Then $[I \setminus (G \times \{l\} \times R)] \cup X$ is an independent dominating set of Δ and

$$\begin{aligned} |[I \setminus (G \times \{l\} \times R)] \cup X| &= |I \setminus (G \times \{l\} \times R)| + |X| \\ &< |I \setminus (G \times \{l\} \times R)| + |I \cap (G \times \{l\} \times R)| \\ &= |[I \setminus (G \times \{l\} \times R)] \cup [I \cap (G \times \{l\} \times R)]| \\ &= |I| \text{ which is a contradiction.} \end{aligned}$$

Therefore, $I \cap (G \times \{l\} \times R)$ is an $i-$set of $(G \times \{l\} \times R, E_l)$, as required. □

Theorem 6.1.3. *If T is an $i-$set of* $\mathrm{Cay}(G \times R, \overline{A})$, *then* $\bigcup_{(t,\gamma) \in T} (\{t\} \times L \times \{\gamma\})$ *is an $i-$set of Δ.*

Proof. Let T be an $i-$set of $\mathrm{Cay}(G \times R, \overline{A})$ and $K = \bigcup_{(t,\gamma) \in T} (\{t\} \times L \times \{\gamma\})$. We will prove that K is an $i-$set of Δ. It is easily seen that K is a nonempty subset of $G \times L \times R$. We first show that K is independent in Δ. Assume that there exist $(t_1,l_1,\lambda_1),(t_2,l_2,\lambda_2) \in K$ such that they are not independent in Δ, that is,

$$((t_1,l_1,\lambda_1),(t_2,l_2,\lambda_2)) \in E(\Delta) \text{ or } ((t_2,l_2,\lambda_2),(t_1,l_1,\lambda_1)) \in E(\Delta).$$

Without loss of generality, we may suppose that $((t_1,l_1,\lambda_1),(t_2,l_2,\lambda_2)) \in E(\Delta)$, that is, $(t_2,l_2,\lambda_2) = (t_1,l_1,\lambda_1)(a,l,\mu) = (t_1 a,l_1,\mu)$ for some $(a,l,\mu) \in A$. Hence $(a,\mu) \in \overline{A}$ and $(t_2,\lambda_2) = (t_1 a,\mu) = (t_1,\lambda_1)(a,\mu)$ which implies that $((t_1,\lambda_1),(t_2,\lambda_2)) \in E(\mathrm{Cay}(G \times R, \overline{A}))$ where $(t_1,\lambda_1),(t_2,\lambda_2) \in T$ which contradicts to the independence of T. Thus K is an independent set of Δ. Next, we will show that K is a dominating set of Δ. Let $(g,m,\rho) \in (G \times L \times R) \setminus K$. Then $(g,\rho) \in (G \times R) \setminus T$. Since T is a dominating set of $\mathrm{Cay}(G \times R, \overline{A})$, we have $((g',\rho'),(g,\rho)) \in E(\mathrm{Cay}(G \times R, \overline{A}))$ for some $(g',\rho') \in T$, that is, there exists $(d,\delta) \in \overline{A}$ such that $(g,\rho) = (g',\rho')(d,\delta)$. Thus $g = g'd$ and $\rho = \delta$. By the definition of $\overline{A}$, there exists $j \in L$ in which $(d,j,\delta) \in A$ which implies that $(g,m,\rho) = (g'd,m,\delta) = (g',m,\rho')(d,j,\delta)$ where $(g',m,\rho') \in K$. Hence $((g',m,\rho'),(g,m,\rho)) \in E(\Delta)$ which leads to the result that K is a dominating set of Δ. We now obtain that K is an independent dominating set of Δ.

Next, we assume that there exists an independent dominating set X of Δ such that

$$i(\Delta) = |X| < |K| = |\bigcup_{(t,\gamma)\in T} (\{t\} \times L \times \{\gamma\})| = |T||L|.$$

Then there exists $l \in L$ such that $|X \cap (G \times \{l\} \times R)| < |T|$. Since X is an $i-$set of Δ, we obtain that $X \cap (G \times \{l\} \times R)$ is an $i-$set of the digraph $(G \times \{l\} \times R, E_l)$ by Theorem 6.1.2. Thus $i(G \times \{l\} \times R, E_l) = |X \cap (G \times \{l\} \times R)|$. By the fact that $(G \times \{l\} \times R, E_l) \cong \text{Cay}(G \times R, \overline{A})$, we can conclude that

$$i(\text{Cay}(G \times R, \overline{A})) = i(G \times \{l\} \times R, E_l) = |X \cap (G \times \{l\} \times R)| < |T|$$

which contradicts to the property of an $i-$set T of $\text{Cay}(G \times R, \overline{A})$. Consequently, we can conclude that $K = \bigcup_{(t,\gamma)\in T} (\{t\} \times L \times \{\gamma\})$ is an $i-$set of Δ. □

Next, we will present the results about an independent domination number $i(\Gamma)$ of a Cayley digraph Γ of a left group $G \times L$ with a connection set A. We recall that

$$[p_1(A)]^{-1} = \{a^{-1} \in G : a \in p_1(A)\}.$$

The sufficient condition for the existence of the independent domination number $i(\Gamma)$ of a Cayley digraph Γ is described as in the following theorem.

Theorem 6.1.4. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. If $p_1(A) = [p_1(A)]^{-1}$, then $i(\Gamma)$ exists.*

Proof. Let A be a connection set of Γ such that $p_1(A) = [p_1(A)]^{-1}$. We first show that every maximal independent set of Γ is also a dominating set of Γ. Let M be a maximal independent set of Γ and $(x, l) \in V(\Gamma) \setminus M$. Assume to the contrary that $((m, q), (x, l)) \notin E(\Gamma)$ for all $(m, q) \in M$. If $((x, l), (x', l')) \notin E(\Gamma)$ for all $(x', l') \in M$, then $M \cup \{(x, l)\}$ is an independent set containing M which is impossible because M is maximal independent. Thus there exists $(y, k) \in M$ such that $((x, l), (y, k)) \in E(\Gamma)$. It follows that there exists $(a, t) \in A$ in which $(y, k) = (x, l)(a, t) = (xa, l)$. We now have $a \in p_1(A) = [p_1(A)]^{-1}$, that is, $a = b^{-1}$ for some $b \in p_1(A)$ and then $yb = (xa)b = x(b^{-1}b) = x$. Since $b \in p_1(A)$, there exists $j \in p_2(A)$ such that $(b, j) \in A$ and we get $(x, l) = (yb, k) = (y, k)(b, j)$. Thus $((y, k), (x, l)) \in E(\Gamma)$ which contradicts to our assumption. Therefore, M is a dominating set of Γ. So we now get that M is an independent dominating set of Γ. Consequently, we can conclude by the definition of $i(\Gamma)$ that $i(\Gamma) \leq |M|$ which guarantees the existence of $i(\Gamma)$, this completes the proof of our assertion. □

The following theorem presents the bounds of an independent domination number $i(\Gamma)$ of Γ under the condition $p_1(A) = [p_1(A)]^{-1}$.

Theorem 6.1.5. *Let A be a connection set of Γ where the identity $e \notin p_1(A)$. If $p_1(A) = [p_1(A)]^{-1}$, then* $\frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lceil \frac{|\langle p_1(A)\rangle|}{|p_1(A)|+1}\right\rceil \leq i(\Gamma) \leq \frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lfloor \frac{|\langle p_1(A)\rangle|}{2}\right\rfloor$.

Proof. Let A be a connection set of Γ such that $p_1(A) = [p_1(A)]^{-1}$ where $e \notin p_1(A)$. Then $i(\Gamma)$ exists by Theorem 6.1.4. Now, we consider the independent domination number of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ which can be considered as a strong subdigraph of Γ. Suppose that $i(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = k$ for some $k \in \mathbb{N}$. Since every vertex of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ dominates other $|p_1(A)|$ vertices, we can conclude that $k(|p_1(A)| + 1) \geq |\langle p_1(A)\rangle|$ which implies that $k \geq \left\lceil \frac{|\langle p_1(A)\rangle|}{|p_1(A)|+1}\right\rceil$. By applying Lemma 2.2.6, we can conclude that

$$i(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}[i(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))] \geq \frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lceil \frac{|\langle p_1(A)\rangle|}{|p_1(A)|+1}\right\rceil.$$

Furthermore, it is uncomplicated to examine that every independent set of the digraph $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ contains at most $\left\lfloor \frac{|\langle p_1(A)\rangle|}{2}\right\rfloor$ elements. Again from Lemma 2.2.6, we can conclude that $i(\Gamma) \leq \frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lfloor \frac{|\langle p_1(A)\rangle|}{2}\right\rfloor$. □

Proposition 6.1.6. *Let A be a connection set of Γ. If $p_1(A)$ satisfies the condition that $x^{-1}y \in p_1(A)$ for any two different elements $x, y \in \langle p_1(A)\rangle$, then $i(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}$.*

Proof. Suppose that the condition holds. Consider the Cayley digraph $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ of a subgroup $\langle p_1(A)\rangle$ with a connection set $p_1(A)$, by our supposition, we obtain that $y = x(x^{-1}y)$ where $x^{-1}y \in p_1(A)$ for all $x, y \in \langle p_1(A)\rangle$ and $x \neq y$. Then $(x, y) \in E(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$. Similarly, we can obtain that $(y, x) \in E(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))$. This means that there exists an arc connecting between two different vertices in $\langle p_1(A)\rangle$. Hence the set $\{v\}$ is an independent dominating set of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ for fixed v in $\langle p_1(A)\rangle$. Thus $i(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = 1$. By Lemma 2.2.6, we conclude that Γ is the disjoint union of $\frac{|G||L|}{|\langle p_1(A)\rangle|}$ strong subdigraphs which each subdigraph is isomorphic to the Cayley digraph $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$. It follows that

$$i(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}[i(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))] = \frac{|G||L|}{|\langle p_1(A)\rangle|}, \text{ as desired.}$$ □

Proposition 6.1.7. *If $A = \{a\} \times T$ where the order of an element a is even and T is a nonempty subset of L, then $i(\Gamma) = \frac{|G||L|}{2}$.*

Proof. Let $A = \{a\} \times T$ be a connection set of Γ such that $|a|$ is even and T is a nonempty subset of L. By the proof of the above proposition, we can conclude that $i(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}[i(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A)))]$. Therefore, we have to consider the independent domination number of $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$. Since $p_1(A) = \{a\}$ and $|a|$ is even, we have that $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$ is the dicycle of even order $|a| = |\langle p_1(A)\rangle|$. In addition, it is not difficult to verify that $i(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = \frac{|\langle p_1(A)\rangle|}{2}$. Consequently, we can conclude that $i(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}\frac{|\langle p_1(A)\rangle|}{2} = \frac{|G||L|}{2}$, as required. □

The next proposition shows the exact value of an independent domination number of Γ with the given connection set A where $p_1(A)$ is a subgroup of G. In addition, we can easily obtain that $i(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = 1$. By applying Lemma 2.2.6, we have the following result.

Proposition 6.1.8. *If $A = K \times T$ where K is a subgroup of G and T is any nonempty subset of L, then $i(\Gamma) = \frac{|G||L|}{|K|}$.*

Here we will show some results about an independent domination number of Cayley digraphs of right groups with their connection sets. Recall that we denote by Λ a Cayley digraph $\mathrm{Cay}(G \times R, A)$ of a right group $G \times R$ with a connection set A.

Theorem 6.1.9. *If $p_2(A) \neq R$, then $i(\Lambda) = |G|(|R| - |p_2(A)|)$.*

Proof. Let A be a connection set of Λ in which $p_2(A) \neq R$. By applying the result about a domination number of Cayley digraphs of right groups as stated in Theorem 4.1.17, we can conclude that $i(\Lambda) \geq |G|(|R| - |p_2(A)|)$. We now consider the subset $G \times (R \setminus p_2(A))$ of $G \times R$. If there exist $(x, \alpha), (y, \beta) \in G \times (R \setminus p_2(A))$ such that $((x, \alpha), (y, \beta)) \in E(\Lambda)$, then $(y, \beta) = (x, \alpha)(a, \lambda)$ for some $(a, \lambda) \in A$, that is, $\beta = \alpha\lambda = \lambda \in p_2(A)$ which is a contradiction. Hence $G \times (R \setminus p_2(A))$ is an independent set of Λ. So we now obtain that $G\times(R\setminus p_2(A))$ is an independent dominating set of Λ. Therefore, $i(\Lambda) \leq |G\times(R\setminus p_2(A))| = |G|(|R| - |p_2(A)|)$. Thus we can conclude that $i(\Lambda) = |G|(|R| - |p_2(A)|)$, as required. □

Theorem 6.1.10. *If $A = \{a\} \times R$ where the order of a is even, then $i(\Lambda) = \frac{|G||R|}{2}$.*

Proof. Suppose that $A = \{a\} \times R$ where $|a|$ is even. We shall show that the set $I := \{a, a^3, a^5, \ldots, a^{k-1}\} \times R$ is an independent dominating set of $\mathrm{Cay}(\langle A\rangle, A)$ where $|a| = k$ for some an even positive integer k. Since $p_1(A) = \{a\}$, it is easy to verify that I is independent. Now, we let $(g, \gamma) \in \langle A\rangle \setminus I$. Thus $g = a^{2t}$ for some $t \in \mathbb{N}$. Since $(g, \gamma) = (a^{2t}, \gamma) = (a^{2t-1}, \gamma)(a, \gamma)$ where $(a, \gamma) \in A$ and $(a^{2t-1}, \gamma) \in I$ that means I is a dominating set of $\mathrm{Cay}(\langle A\rangle, A)$. Hence I is an independent dominating set of $\mathrm{Cay}(\langle A\rangle, A)$ which implies that $i(\mathrm{Cay}(\langle A\rangle, A)) \leq |I|$. We now assume that there exists an independent dominating set X of $\mathrm{Cay}(\langle A\rangle, A)$ such that $|X| < |I|$. Define the subset X' of $\langle A\rangle$ as follows:

$$X' = \{(x', \delta') \in \langle A\rangle : ((x, \delta), (x', \delta')) \in E(\mathrm{Cay}(\langle A\rangle, A)) \text{ for some } (x, \delta) \in X\}.$$

Thus $X \cap X' = \emptyset$ by the independence of X and we have $X \cup X' = \langle A \rangle$ since X is the dominating set of $\mathrm{Cay}(\langle A \rangle, A)$. Hence

$$|\langle A \rangle| = |X \cup X'| = |X| + |X'| = |X| + \tfrac{k}{2}|R| = |X| + |I| < |I| + |I| = 2|I|.$$

Since $p_2(A) = R$ and by Lemma 2.2.8, we obtain that

$$|\langle A \rangle| < 2|I| = 2(\tfrac{k}{2}|R|) = k|R| = |\langle a \rangle||R| = |\langle p_1(A) \rangle||R| = |\langle p_1(A) \rangle \times R| = |\langle A \rangle|$$

which is a contradiction. Therefore, $i(\mathrm{Cay}(\langle A \rangle, A)) = |I| = \frac{k}{2}|R|$. Since Λ is the disjoint union of $\frac{|G|}{|\langle p_1(A) \rangle|}$ strong subdigraphs which each subdigraph is isomorphic to $\mathrm{Cay}(\langle A \rangle, A)$ as stated in Lemma 2.2.7, we can conclude that

$$i(\Lambda) = \tfrac{|G|}{|\langle p_1(A) \rangle|}(i(\mathrm{Cay}(\langle A \rangle, A))) = \tfrac{|G|}{|\langle p_1(A) \rangle|}(\tfrac{k}{2}|R|) = \tfrac{|G||R|}{2}.$$

Hence our assertion is completely proved. □

Theorem 6.1.11. *If $A = K \times R$ where K is a subgroup of G, then $i(\Lambda) = \frac{|G|}{|K|}$.*

Proof. Let A be a connection set of Λ such that $A = K \times R$ where K is a subgroup of G. By Lemma 2.2.8, we can conclude that

$$\langle A \rangle = \langle p_1(A) \rangle \times p_2(A) = \langle K \rangle \times R = K \times R = A.$$

For any two elements (x, α) and (y, β) in $\langle A \rangle$, we have $(y, \beta) = (x, \alpha)(x^{-1}y, \beta)$ where $(x^{-1}y, \beta) \in K \times R = A$. This means that there exists an arc in $\mathrm{Cay}(\langle A \rangle, A)$ connecting between two different vertices of $\mathrm{Cay}(\langle A \rangle, A)$. Thus we can use only one vertex in $\langle A \rangle$ for dominating other vertices in $\langle A \rangle$ which implies that $i(\mathrm{Cay}(\langle A \rangle, A)) = 1$. By Lemma 2.2.7, we obtain that $i(\Lambda) = \frac{|G|}{|\langle p_1(A) \rangle|}(i(\mathrm{Cay}(\langle A \rangle, A))) = \frac{|G|}{|K|}(i(\mathrm{Cay}(\langle A \rangle, A))) = \frac{|G|}{|K|}$. □

6.2 Weakly Independent Domination Number

This section provides some results of the weakly independent domination number of Cayley digraphs of rectangular groups, left groups, and right groups, respectively.

For the weakly independent domination number $i_w(\Delta)$ of a Cayley digraph Δ of a rectangular group $G \times L \times R$ with a connection set A, we can similarly follow those results from Theorems 6.1.1, 6.1.2 and 6.1.3 to obtain the following results.

Theorem 6.2.1. *If $\overline{A} = \{(a, \alpha) \in G \times R : (a, l, \alpha) \in A$ for some $l \in L\}$, then $i_w(\Delta) = |L|(i_w(\mathrm{Cay}(G \times R, \overline{A})))$.*

Theorem 6.2.2. *If I is an i_w-set of Δ, then $I \cap (G \times \{l\} \times R)$ is an i_w-set of the digraph $(G \times \{l\} \times R, E_l)$ for all $l \in L$.*

Theorem 6.2.3. *If T is an i_w-set of* $\mathrm{Cay}(G \times R, \overline{A})$*, then* $\bigcup_{(t,\gamma)\in T} (\{t\} \times L \times \{\gamma\})$ *is an i_w-set of Δ.*

Here we will present some results about a weakly independent domination number $i_w(\Gamma)$ of a Cayley digraph Γ of a left group $G \times L$ with a connection set A.

Proposition 6.2.4. *If $p_1(A)$ is a subgroup of G, then* $i_w(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}$.

Proof. Let A be a connection set of Γ such that $p_1(A)$ is a subgroup of a group G. By applying the similar argument in the proof of Proposition 6.1.6, we can conclude that $i_w(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = 1$. Hence $i_w(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}$ by using Lemma 2.2.6. □

Proposition 6.2.5. *If $p_1(A) = \{a, a^{-1}\}$ for some $a \in G$ and a is not the identity of G, then* $i_w(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|} \left\lceil \frac{|\langle p_1(A)\rangle|}{3} \right\rceil$.

Proof. Let A be a connection set of Γ such that $p_1(A) = \{a, a^{-1}\}$ where a is not the identity of a group G. Consider $\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))$, we obtain that for each $g \in \langle p_1(A)\rangle$, g can dominate other two vertices ga and ga^{-1}. Then it is not difficult to verify that $i_w(\mathrm{Cay}(\langle p_1(A)\rangle, p_1(A))) = \left\lceil \frac{|\langle p_1(A)\rangle|}{3} \right\rceil$. Therefore, we can certainly conclude that $i_w(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|} \left\lceil \frac{|\langle p_1(A)\rangle|}{3} \right\rceil$ which completes the proof. □

Next, we will present the results about a weakly independent domination number $i_w(\Lambda)$ of a Cayley digraph Λ of a right group $G \times R$ with respect to a connection set A.

Theorem 6.2.6. *If $p_2(A) \neq R$, then $i_w(\Lambda) = |G|(|R| - |p_2(A)|)$.*

Proof. By using the similar argument of the proof of Theorem 6.1.9, we can conclude that $i_w(\Lambda) = |G|(|R| - |p_2(A)|)$, as desired. □

Theorem 6.2.7. *If $p_1(A) = G$, $p_2(A) = R$ and $|A| = |R|$, then $i_w(\Lambda) = |G|$.*

Proof. Let A be a connection set of Λ such that $p_1(A) = G$, $p_2(A) = R$ and $|A| = |R|$. Consider $\gamma \in R$ in which $(e, \gamma) \in A$ where e is the identity of a group G, we obtain from the proof of Theorem 4.1.22 that $G \times \{\gamma\}$ is a dominating set of Λ such that the domination number of Λ equals $|G \times \{\gamma\}|$. Since all arcs of a strong subdigraph of Λ induced by $G \times \{\gamma\}$ are loops, every two vertices in $G \times \{\gamma\}$ are weakly independent. So we can conclude that $G \times \{\gamma\}$ is an i_w-set that means $i_w(\Lambda) = |G \times \{\gamma\}| = |G|$, an assertion of the theorem is proved. □

Proposition 6.2.8. *If $A = \{a\} \times R$ where $|a| = 2$, then* $i_w(\Lambda) = \frac{|G||R|}{2}$.

Proof. Let $A = \{a\} \times R$ be a connection set of Λ such that $|a| = 2$. We can observe by Lemma 2.2.7 that Λ is the disjoint union of $\frac{|G|}{|\langle a\rangle|}$ strong subdigraphs which each strong subdigraph is isomorphic to $\text{Cay}(\langle A\rangle, A)$. Since $\langle A\rangle = \langle p_1(A)\rangle \times p_2(A) = \langle a\rangle \times R$ as shown in Lemma 2.2.8, we obtain that $\{a\} \times R$ is a weakly independent dominating set of $\text{Cay}(\langle A\rangle, A)$. Let Y be any weakly independent dominating set of $\text{Cay}(\langle A\rangle, A)$. For each $(y, \mu) \in Y$, we can obtain that $\{ya\} \times R$ must be dominated by (y, μ). Thus any element of $\{ya\} \times R$ can not belong to Y since Y is weakly independent. Hence Y precisely contains all elements of $\{y\} \times R$ which makes us conclude that

$$i_w(\text{Cay}(\langle A\rangle, A)) = |Y| = |\{y\} \times R| = |R|.$$

Therefore, $i_w(\Lambda) = \frac{|G|}{|\langle a\rangle|}(i_w(\text{Cay}(\langle A\rangle, A))) = \frac{|G||R|}{2}$, as required. □

Theorem 6.2.9. *If $A = K \times R$ where K is a subgroup of G, then $i_w(\Lambda) = \frac{|G|}{|K|}$.*

Proof. Similar to the proof of Theorem 6.1.11. □

6.3 Dipath Independent Domination Number

In this section, some results about a dipath independent domination number of Cayley digraphs of rectangular groups with respect to their connection sets are presented. For convenience, we will recall that Δ is the Cayley digraph $\text{Cay}(G \times L \times R, A)$ of a rectangular group $G \times L \times R$ with a connection set A.

Actually, we can obtain the corresponding results from Theorems 6.1.1, 6.1.2 and 6.1.3 for a dipath independent domination number of Cayley digraphs of rectangular groups with arbitrary connection sets as follows.

Theorem 6.3.1. *If $\overline{A} = \{(a, \alpha) \in G \times R : (a, l, \alpha) \in A$ for some $l \in L\}$, then $i_p(\Delta) = |L|(i_p(\text{Cay}(G \times R, \overline{A})))$.*

Theorem 6.3.2. *If I is an i_p-set of Δ, then $I \cap (G \times \{l\} \times R)$ is an i_p-set of the digraph $(G \times \{l\} \times R, E_l)$ for all $l \in L$.*

Theorem 6.3.3. *If T is an i_p-set of $\text{Cay}(G \times R, \overline{A})$, then $\bigcup_{(t,\gamma)\in T} (\{t\} \times L \times \{\gamma\})$ is an i_p-set of Δ.*

Next, we will show other results for the dipath independent domination number of Cayley digraphs of rectangular groups with their connection sets.

Theorem 6.3.4. *Let Δ be a Cayley digraph of a rectangular group $G \times L \times R$ with a connection set A. If $p_3(A) \neq R$, then $i_p(\Delta) = |G||L|(|R| - |p_3(A)|)$.*

Proof. Let A be a connection set of Δ such that $p_3(A) \neq R$. Since Δ is the disjoint union of $|L|$ strong subdigraphs which each subdigraph is isomorphic to $\text{Cay}(G \times R, \overline{A})$ where $\overline{A} = \{(a, \gamma) \in G \times R : (a, l, \gamma) \in A \text{ for some } l \in L\}$ as stated in Theorem 3.1.1, we need to consider the dipath independent domination number of that Cayley digraph $\text{Cay}(G \times R, \overline{A})$. From $p_3(A) \neq R$, it follows that $p_2(\overline{A}) \neq R$. We can conclude by applying Theorem 4.1.17 that $G \times (R \setminus p_2(\overline{A}))$ is a dominating set and its cardinality equals the domination number of $\text{Cay}(G \times R, \overline{A})$. In addition, we can verify that $G \times (R \setminus p_2(\overline{A}))$ is a dipath independent set of $\text{Cay}(G \times R, \overline{A})$, clearly. So we now have that $G \times (R \setminus p_2(\overline{A}))$ is a dipath independent dominating set of $\text{Cay}(G \times R, \overline{A})$. Therefore, $G \times (R \setminus p_2(\overline{A}))$ is an i_p-set of $\text{Cay}(G \times R, \overline{A})$ that means $i_p(\text{Cay}(G \times R, \overline{A})) = |G|(|R| - |p_2(\overline{A})|)$. From the fact that $p_2(\overline{A}) = p_3(A)$, we get that $i_p(\Delta) = |G||L|(|R| - |p_3(A)|)$, certainly. □

We now give the necessary and sufficient conditions for the existence of a dipath independent domination number of $\Delta := \text{Cay}(G \times L \times R, A)$ where the connection set A satisfies the condition that $p_3(A) = R$.

Theorem 6.3.5. *If $p_3(A) = R$, then the following statements are equivalent:*
(1). $i_p(\Delta)$ *exists;* (2). $\overline{A} = \langle p_1(A) \rangle \times R$; (3). $i_p(\Delta) = \frac{|G||L|}{|\langle p_1(A) \rangle|}$.

Proof. Let A be a connection set of Δ such that $p_3(A) = R$.

(1) $\Rightarrow$ (2) : Suppose that $i_p(\Delta)$ exists, that is, $i_p(\Delta) = |X|$ for some an i_p-set X of Δ. We first let $(a, \gamma) \in \overline{A}$. Hence there exists $l \in L$ in which $(a, l, \gamma) \in A$ which implies that $a \in p_1(A) \subseteq \langle p_1(A) \rangle$, whence $(a, \gamma) \in \langle p_1(A) \rangle \times R$. Next, we need to give $(b, \tau) \in \langle p_1(A) \rangle \times R$ for proving the reverse containment. Since $p_3(A) = R$, we have by Theorem 3.2.1 that $\text{Cay}(\langle \overline{A} \rangle, \overline{A})$ is strongly connected, that is, there exists a dipath joining between any two vertices in $\text{Cay}(\langle \overline{A} \rangle, \overline{A})$. We obviously obtain that $p_2(\overline{A}) = R$. By Lemma 2.2.8 and Proposition 3.2.2, we can obtain that $\langle \overline{A} \rangle = \langle p_1(\overline{A}) \rangle \times R = \langle p_1(A) \rangle \times R = p_1(\langle A \rangle) \times R$. For each $l \in L$, we also get that $p_1(\langle A \rangle) \times R$ is isomorphic to $p_1(\langle A \rangle) \times \{l\} \times R$. We will denote $p_1(\langle A \rangle) \times \{l\} \times R$ by $\langle A \rangle_l$. Thus the strong subdigraph of Δ induced by $\langle A \rangle_l$ can be considered to be strongly connected. Since X is a dipath independent dominating set of Δ, we obtain that $|X \cap \langle A \rangle_l| = 1$. We may assume that $X \cap \langle A \rangle_l = \{(x, l, \lambda)\}$ for some $(x, l, \lambda) \in \langle A \rangle_l$, that is, $x \in p_1(\langle A \rangle)$. Applying Proposition 3.2.2 gives $(xb, l, \tau) \in \langle A \rangle_l$. By the definition of X, we conclude that (xb, l, τ) must be dominated by (x, l, λ) that means there exists $(y, k, \delta) \in A$ such that $(xb, l, \tau) = (x, l, \lambda)(y, k, \delta) = (xy, l, \delta)$. Thus $xb = xy$ and $\tau = \delta$ which implies by the cancellation law that $(b, \tau) = (y, \delta) \in \overline{A}$ since $(y, k, \delta) \in A$. So we can conclude that $\overline{A} = \langle p_1(A) \rangle \times R$, as required.

(2) $\Rightarrow$ (3) : Assume that $\overline{A} = \langle p_1(A)\rangle \times R$. We first prove that there exists an arc connecting between any two vertices of $\mathrm{Cay}(\langle\overline{A}\rangle, \overline{A})$. Let $(a,\gamma), (b,\delta) \in \langle\overline{A}\rangle$. Thus $(a,\gamma) = (a_1,\gamma_1)(a_2,\gamma_2)\cdots(a_m,\gamma_m)$ and $(b,\delta) = (b_1,\delta_1)(b_2,\delta_2)\cdots(b_n,\delta_n)$ for some $(a_1,\gamma_1),(a_2,\gamma_2),\ldots,(a_m,\gamma_m),(b_1,\delta_1),(b_2,\delta_2),\ldots,(b_n,\delta_n) \in \overline{A}$. Hence $a = a_1a_2\cdots a_m$ and $b = b_1b_2\cdots b_n$ in which $a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_n \in p_1(\overline{A}) = \langle p_1(A)\rangle$. Therefore, we can directly observe that

$$a^{-1}b = (a_1a_2\cdots a_m)^{-1}b_1b_2\cdots b_n = a_m^{-1}a_{m-1}^{-1}\cdots a_2^{-1}a_1^{-1}b_1b_2\cdots b_n \in \langle p_1(A)\rangle.$$

Hence $(a^{-1}b,\delta) \in \langle p_1(A)\rangle \times R = \overline{A}$ such that $(b,\delta) = (a,\gamma)(a^{-1}b,\delta)$ which leads to the fact that $((a,\gamma),(b,\delta)) \in E(\mathrm{Cay}(\langle\overline{A}\rangle,\overline{A}))$. So we can conclude that $\{(x,\tau)\}$, $(x,\tau) \in \langle\overline{A}\rangle$, is a dipath independent dominating set of $\mathrm{Cay}(\langle\overline{A}\rangle,\overline{A})$ which follows that $i_p(\mathrm{Cay}(\langle\overline{A}\rangle,\overline{A})) = 1$. By Theorem 3.1.1, we get that Δ is the disjoint union of $|L|$ strong subdigraphs which each subdigraph is isomorphic to $\mathrm{Cay}(G\times R,\overline{A})$. In addition, we have by Lemma 2.2.7 that $\mathrm{Cay}(G\times R,\overline{A})$ is the disjoint union of $\frac{|G|}{|\langle p_1(\overline{A})\rangle|}$ strong subdigraphs which each subdigraph is isomorphic to $\mathrm{Cay}(\langle\overline{A}\rangle,\overline{A})$. Therefore, we obtain that

$$i_p(\Delta) = \frac{|G||L|}{|\langle p_1(\overline{A})\rangle|}[i_p(\mathrm{Cay}(\langle\overline{A}\rangle,\overline{A}))] = \frac{|G||L|}{|\langle p_1(\overline{A})\rangle|}.$$

It is easy to see from our assumption that $\langle p_1(\overline{A})\rangle = \langle p_1(A)\rangle$ which certainly follows that $i_p(\Delta) = \frac{|G||L|}{|\langle p_1(A)\rangle|}$, proving the assertion. It is obvious for (3) $\Rightarrow$ (1). □

6.4 Weakly Dipath Independent Domination Number

This section shows some results about a weakly dipath independent domination number of Cayley digraphs of rectangular groups with respect to their connection sets. We will recall that Δ is the Cayley digraph $\mathrm{Cay}(G\times L\times R, A)$ of a rectangular group $G\times L\times R$ with a connection set A.

The following three theorems are analogous results of Theorems 6.3.1, 6.3.2 and 6.3.3 which describe the results for a weakly dipath independent domination number of Cayley digraphs of rectangular groups relative to their connection sets.

Theorem 6.4.1. *If* $\overline{A} = \{(a,\alpha) \in G\times R : (a,l,\alpha) \in A$ *for some* $l \in L\}$, *then* $i_{wp}(\Delta) = |L|(i_{wp}(\mathrm{Cay}(G\times R,\overline{A})))$.

Theorem 6.4.2. *If* I *is an* $i_{wp}-$*set of* Δ, *then* $I \cap (G\times\{l\}\times R)$ *is an* $i_{wp}-$*set of the digraph* $(G\times\{l\}\times R, E_l)$ *for all* $l \in L$.

Theorem 6.4.3. *If* T *is an* $i_{wp}-$*set of* $\mathrm{Cay}(G\times R,\overline{A})$, *then* $\bigcup_{(t,\gamma)\in T}(\{t\}\times L\times\{\gamma\})$ *is an* $i_{wp}-$*set of* Δ.

The following two theorems are stated to show the results for a weakly dipath independent domination number of Cayley digraphs of rectangular groups with respect to their connection sets. The last theorem gives the sufficient condition for a weakly dipath independent domination number and a dipath independent domination number of Δ to be equal.

Theorem 6.4.4. *If* $p_3(A) \neq R$, *then* $i_{wp}(\Delta) = |G||L|(|R| - |p_3(A)|)$.

Proof. It is easy to verify that the i_p-set mentioned in the proof of Theorem 6.3.4 is an $i_{wp}-$set of Δ which completely leads to our assertion. □

Theorem 6.4.5. *If* $p_3(A) = R$, *then* $i_{wp}(\Delta) = i_p(\Delta)$.

Proof. Assume that $p_3(A) = R$. By Theorem 3.2.1, we obtain that $\mathrm{Cay}(\langle \overline{A} \rangle, \overline{A})$ is strongly connected, that is, there exists a dipath connecting between any two different vertices of $\mathrm{Cay}(\langle \overline{A} \rangle, \overline{A})$. Therefore, we can consider the property of the weakly dipath independent domination number of the Cayley digraph Δ to be the same as the property of the dipath independent domination number of Δ which concludes the result of our theorem. □