CHAPTER 7

Endomorphisms on Cayley Digraphs of Rectangular Groups

The purpose of this chapter is to present the results for endomorphisms on Cayley digraphs of some rectangular groups related to the according connection sets. Let (V, E) be a digraph with a vertex set V and an arc set E. Recall that a function $f : (V, E) \to (V, E)$ is called an endomorphism on (V, E) if for each $(x, y) \in E$ implies $(f(x), f(y)) \in E$ as well. We divide this chapter into three sections. In the first section, we give some results for endomorphisms on Cayley digraphs of rectangular groups with their connection sets. The remaining sections are studied about endomorphisms on Cayley digraphs of left groups and right groups, respectively.

In order to study the structure of endomorphisms on Cayley digraphs of rectangular groups, we need to prescribe some notations used in what follows. For any digraph Ω, let us denote by this notation (Γ, E) the strong subdigraph of Ω induced by Γ in which a vertex set is Γ and an arc set is E. For each a function $f : \Omega_1 \longrightarrow \Omega_2$ from a digraph Ω_1 to a digraph Ω_2 and any subdigraph Σ of Ω_1, we mention $f(\Sigma)$ as a strong subdigraph $(f(\Sigma), E')$ of Ω_2 induced by $f(\Sigma)$.

7.1 Endomorphisms on Cayley Digraphs of Rectangular Groups

We now study the endomorphisms on a Cayley digraph Δ of a rectangular group $G \times L \times R$ with respect to a connection set A. Before we give those results, we first define the useful function which is used in the sequel. Let $f : \Delta \to \Delta$ be a function from Δ into Δ itself and $l \in L$. For each $\alpha \in R$, we define $\Phi_{l\alpha} : \mathrm{Cay}(G, p_1(A)) \to \mathrm{Cay}(G, p_1(A))$ by

$$\Phi_{l\alpha}(a) = b \text{ if and only if there exist } t \in L \text{ and } \beta \in R$$
$$\text{such that } f(a, l, \alpha) = (b, t, \beta) \text{ for all } a \in G.$$

It is easy to verify that $\Phi_{l\alpha}$ is well-defined. Some results for endomorphisms on a Cayley digraph $\Delta := \mathrm{Cay}(G \times L \times R, A)$ are obtained such that the connection set A is in the form of a direct product $K \times P \times T$ where $K \subseteq G$, $P \subseteq L$ and $T \subseteq R$.

Theorem 7.1.1. *Let $A = K \times P \times T$ be a connection set of Δ and $l \in L$. If $f : \Delta \to \Delta$, then $f \in \mathrm{End}(\Delta)$ if and only if the following statements hold:*

(1). *$f(b\langle K\rangle \times \{l\} \times R, E_l)$ is a subdigraph of $(c\langle K\rangle \times \{t\} \times R, E_t)$ for some $t \in L$, $c \in G$ and for all $b \in G$;*

(2). *$\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$ for all $\alpha \in T$;*

(3). *for each $g \in K$ and $a \in G$, there exists $g_a \in K$ such that*

$$f(ag^{-1}, l, \theta) \in \begin{cases} \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times T & \text{if } \theta \in T, \\ \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times R & \text{if } \theta \in R \setminus T \end{cases}$$

for all $\lambda \in T$ and for some $u \in L$.

Proof. Let $A = K \times P \times T$ be a connection set of Δ, $l \in L$ and $f : \Delta \to \Delta$.
($\Rightarrow$) Assume that $f \in \mathrm{End}(\Delta)$.

Let $b \in G$. We prove that $f(b\langle K\rangle \times \{l\} \times R, E_l)$ is a subdigraph of $(c\langle K\rangle \times \{t\} \times R, E_t)$ for some $t \in L$ and $c \in G$. Let $(g, p, r), (h, k, s) \in V(f(b\langle K\rangle \times \{l\} \times R, E_l))$ be such that $(g, p, r) \in V(c\langle K\rangle \times \{t\} \times R, E_t)$ and $(h, k, s) \in V(d\langle K\rangle \times \{u\} \times R, E_u)$ for some $t, u \in L$ and $c, d \in G$. Thus $p = t$ and $k = u$ and hence $(g, t, r) = (g, p, r) = f(g', p', r')$ for some $(g', p', r') \in b\langle K\rangle \times \{l\} \times R$ and $(h, u, s) = (h, k, s) = f(h', k', s')$ for some $(h', k', s') \in b\langle K\rangle \times \{l\} \times R$. Then $p' = l = k'$. Since $((g', l, r'), (g', l, r')(a, i, x)) \in E(\Delta)$ where $(a, i, x) \in A$ and $f \in \mathrm{End}(\Delta)$, we have $(f(g', l, r'), f(g'a, l, x)) \in E(\Delta)$. Similarly, we conclude that $((h', l, s'), (h', l, s')(a, i, x)) \in E(\Delta)$ implies $(f(h', l, s'), f(h'a, l, x)) \in E(\Delta)$. From $g', h' \in b\langle K\rangle$ and $a \in K$, we get $g'a, h'a \in b\langle K\rangle$. Consider the strong subdigraph $(b\langle K\rangle \times \{l\} \times \{x\}, E_{lx})$ of Δ which is isomorphic to $\mathrm{Cay}(\langle K\rangle, K)$, we obtain that there exists a dipath connecting between $(g'a, l, x)$ and $(h'a, l, x)$, say M. We may assume that

$$M := (g'a, l, x), m_1, m_2, \ldots, m_q, (h'a, l, x)$$

where $m_j \in b\langle K\rangle \times \{l\} \times \{x\}$ and $j = 1, 2, \ldots, q$. Since $f \in \mathrm{End}(\Delta)$, we have

$$f(g'a, l, x), f(m_1), f(m_2), \ldots, f(m_q), f(h'a, l, x)$$

is a diwalk in Δ. Hence there exists a semi-diwalk connecting between $f(g', p', r')$ and $f(h', k', s')$. Since two digraphs $(c\langle K\rangle \times \{t\} \times R, E_t)$ and $(d\langle K\rangle \times \{u\} \times R, E_u)$ are maximal semi-connected subdigraphs of Δ, we can conclude that

$$(c\langle K\rangle \times \{t\} \times R, E_t) = (d\langle K\rangle \times \{u\} \times R, E_u),$$

that is, $t = u$. Therefore, $V(f(b\langle K\rangle \times \{l\} \times R, E_l)) \subseteq V(c\langle K\rangle \times \{t\} \times R, E_t)$. We now let $((g_1, p_1, r_1), (g_2, p_2, r_2)) \in E(f(b\langle K\rangle \times \{l\} \times R, E_l))$. We then obtain that

$$(g_1, p_1, r_1), (g_2, p_2, r_2) \in V(f(b\langle K\rangle \times \{l\} \times R, E_l)) \subseteq V(c\langle K\rangle \times \{t\} \times R, E_t).$$

Thus $((g_1, p_1, r_1), (g_2, p_2, r_2)) \in E(c\langle K\rangle \times \{t\} \times R, E_t)$ since $(c\langle K\rangle \times \{t\} \times R, E_t)$ is a strong subdigraph of Δ. Consequently, the digraph $f(b\langle K\rangle \times \{l\} \times R, E_l)$ is a subdigraph of $(c\langle K\rangle \times \{t\} \times R, E_t)$, as required.

We next prove that $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$ for all $\alpha \in T$. Let $\alpha \in T$ and $(x, y) \in E(\mathrm{Cay}(G, K))$. Thus $y = xa$ for some $a \in K$. Assume that $\Phi_{l\alpha}(x) = u$ and $\Phi_{l\alpha}(y) = v$ for some $u, v \in G$. Then $f(x, l, \alpha) = (u, k, \beta)$ and $f(y, l, \alpha) = (v, q, \gamma)$ for some $k, q \in L$ and $\beta, \gamma \in R$. Since $((x, l, \alpha), (y, l, \alpha)) = ((x, l, \alpha), (xa, l, \alpha)) = ((x, l, \alpha), (x, l, \alpha)(a, p, \alpha)) \in E(\Delta)$ where $(a, p, \alpha)) \in A$ and $f \in \mathrm{End}(\Delta)$, we have $(f(x, l, \alpha), f(y, l, \alpha)) \in E(\Delta)$, that is, $f(y, l, \alpha) = f(x, l, \alpha)(b, m, \lambda)$ for some $(b, m, \lambda) \in A$. Hence $(v, q, \gamma) = f(y, l, \alpha) = f(x, l, \alpha)(b, m, \lambda) = (u, k, \beta)(b, m, \lambda) = (ub, k, \lambda)$. We obtain that $v = ub$ that means $\Phi_{l\alpha}(y) = v = ub = \Phi_{l\alpha}(x)b$ where $b \in K$. Therefore, $(\Phi_{l\alpha}(x), \Phi_{l\alpha}(y)) \in E(\mathrm{Cay}(G, K))$ which implies that $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$.

In order to prove the statement (3), we now let $\lambda \in T$ and $\theta \in R$. For each $g \in K$ and $a \in G$, consider $(ag^{-1}, l, \theta) \in V(\Delta)$. Since $(a, l, \lambda) = (ag^{-1}, l, \theta)(g, p, \lambda)$ where $(g, p, \lambda) \in A$, we get $((ag^{-1}, l, \theta), (a, l, \lambda)) \in E(\Delta)$. Because $f \in \mathrm{End}(\Delta)$, we obtain that $(f(ag^{-1}, l, \theta), f(a, l, \lambda)) \in E(\Delta)$. We may assume that $f(ag^{-1}, l, \theta) = (h, u, \delta)$ for some $(h, u, \delta) \in V(\Delta)$. Then there exists $(g_a, i, \mu) \in A$ such that

$$f(a, l, \lambda) = f(ag^{-1}, l, \theta)(g_a, i, \mu) = (h, u, \delta)(g_a, i, \mu) = (hg_a, u, \mu).$$

Hence $\Phi_{l\lambda}(a) = hg_a$. Thus $f(ag^{-1}, l, \theta) = (h, u, \delta) = (hg_a g_a^{-1}, u, \delta) = (\Phi_{l\lambda}(a)g_a^{-1}, u, \delta)$. If $\theta \in T$, then $(g, p, \theta) \in A$. Since $(ag^{-1}, l, \theta) = (ag^{-2}, l, \theta)(g, p, \theta)$, we obtain that $((ag^{-2}, l, \theta), (ag^{-1}, l, \theta)) \in E(\Delta)$ and hence $(f(ag^{-2}, l, \theta), f(ag^{-1}, l, \theta)) \in E(\Delta)$ because $f \in \mathrm{End}(\Delta)$. Suppose that $f(ag^{-2}, l, \theta) = (c, e, \varepsilon)$ for some $(c, e, \varepsilon) \in V(\Delta)$. Hence

$$f(ag^{-1}, l, \theta) = f(ag^{-2}, l, \theta)(m, w, \eta) = (c, e, \varepsilon)(m, w, \eta) = (cm, e, \eta)$$

for some $(m, w, \eta) \in A$. We can conclude that $(\Phi_{l\lambda}(a)g_a^{-1}, u, \delta) = (cm, e, \eta)$ which leads to $\delta = \eta \in T$. Therefore,

$$f(ag^{-1}, l, \theta) \in \begin{cases} \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times T & \text{if } \theta \in T, \\ \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times R & \text{if } \theta \in R \setminus T. \end{cases}$$

($\Leftarrow$) We now suppose that the conditions hold. Let $((a, l, \rho), (b, j, \lambda)) \in E(\Delta)$. Thus there exists $(k, p, t) \in A$ such that $(b, j, \lambda) = (a, l, \rho)(k, p, t) = (ak, l, t)$. Then $b = ak$, $j = l$ and $\lambda = t \in T$. Since $(a, b) = (a, ak) \in E(\mathrm{Cay}(G, K))$ and $\Phi_{l\lambda} \in \mathrm{End}(\mathrm{Cay}(G, K))$, we get that $(\Phi_{l\lambda}(a), \Phi_{l\lambda}(b)) \in E(\mathrm{Cay}(G, K))$. Hence $\Phi_{l\lambda}(b) = \Phi_{l\lambda}(a)c$ for some $c \in K$. By our supposition, there exist $u \in L,\ \mu \in R$ and $q \in K$ in which

$$f(a, l, \rho) = f(akk^{-1}, l, \rho) = (\Phi_{l\lambda}(ak)q^{-1}, u, \mu) = (\Phi_{l\lambda}(b)q^{-1}, u, \mu) = (\Phi_{l\lambda}(a)cq^{-1}, u, \mu).$$

By the definition of $\Phi_{l\lambda}$, there exist $m \in L$ and $\omega \in R$ such that $f(b, j, \lambda) = f(b, l, \lambda) = (\Phi_{l\lambda}(b), m, \omega)$. Since $\lambda = t \in T$, we obtain by our supposition that there exists $s \in K$ such that $f(b, j, \lambda) = f(bnn^{-1}, j, \lambda) = (\Phi_{j\lambda}(bn)s^{-1}, v, \xi)$ for some $n \in K,\ v \in L$ and $\xi \in T$. Thus $\omega = \xi \in T$ and hence $f(b, j, \lambda) = (\Phi_{l\lambda}(b), m, \omega) = (\Phi_{l\lambda}(a)c, m, \omega)$. Since $j = l$ and $((a, l, \rho), (b, j, \lambda)) \in E(\Delta)$, we gain that $(a, l, \rho), (b, j, \lambda) \in V(g\langle K\rangle \times \{l\} \times R, E_l)$ for some $g \in G$. We obtain that $f(a, l, \rho), f(b, j, \lambda) \in V(f(g\langle K\rangle \times \{l\} \times R, E_l))$. Since $f(g\langle K\rangle \times \{l\} \times R, E_l)$ is a subdigraph of $(h\langle K\rangle \times \{p\} \times R, E_p)$ for some $h \in G$ and $p \in L$, both $f(a, l, \rho)$ and $f(b, j, \lambda)$ must belong to the vertex set of the same strong subdigraph of Δ. From $f(a, l, \rho) = (\Phi_{l\lambda}(a)cq^{-1}, u, \mu)$ and $f(b, j, \lambda) = (\Phi_{l\lambda}(b), m, \omega)$, we can conclude that $f(a, l, \rho), f(b, j, \lambda) \in V(d\langle K\rangle \times \{u\} \times R, E_u)$ for some $d \in G$ that means $m = u$. For fixed $y \in P$, we have $(q, y, \omega) \in K \times P \times T = A$ and then

$$\begin{aligned} f(b, j, \lambda) &= (\Phi_{l\lambda}(a)c, m, \omega) = (\Phi_{l\lambda}(a)c, u, \omega) \\ &= (\Phi_{l\lambda}(a)cq^{-1}, u, \mu)(q, y, \omega) \\ &= f(a, l, \rho)(q, y, \omega). \end{aligned}$$

Hence $(f(a, l, \rho), f(b, j, \lambda)) \in E(\Delta)$ which leads to $f \in \mathrm{End}(\Delta)$, as required. □

Now, we will illustrate an example of an endomorphism of a Cayley digraph of a rectangular group with a connection set A as stated in Theorem 7.1.1 and indicate that the endomorphism satisfies three conditions as shown in Theorem 7.1.1.

Example 7.1.2. Let $\Delta = \mathrm{Cay}(\mathbb{Z}_3 \times \{l, k\} \times \{\alpha, \beta\}, A)$ where $A = \{1\} \times \{l\} \times \{\alpha\}$.

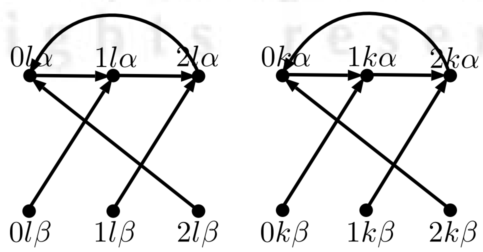

Figure 7.1: $\mathrm{Cay}(\mathbb{Z}_3 \times \{l, k\} \times \{\alpha, \beta\}, \{(1, l, \alpha)\})$.

We obtain that

$$f = \begin{pmatrix} 0l\alpha & 1l\alpha & 2l\alpha & 0k\alpha & 1k\alpha & 2k\alpha & 0l\beta & 1l\beta & 2l\beta & 0k\beta & 1k\beta & 2k\beta \\ 1k\alpha & 2k\alpha & 0k\alpha & 2l\alpha & 0l\alpha & 1l\alpha & 1k\beta & 2k\alpha & 0k\alpha & 2l\beta & 0l\beta & 1l\beta \end{pmatrix} \in \mathrm{End}(\Delta).$$

From $p_1(A) = \{1\}$, we have $\langle p_1(A)\rangle = \{0, 1, 2\}$. Consider

$$f(\langle p_1(A)\rangle \times \{l\} \times \{\alpha, \beta\}) = \{1k\alpha, 2k\alpha, 0k\alpha, 1k\beta\},$$

we get that $f(\langle p_1(A)\rangle \times \{l\} \times \{\alpha, \beta\}, E_l)$ is the subdigraph of $(\langle p_1(A)\rangle \times \{k\} \times \{\alpha, \beta\}, E_k)$ as shown in Figure 7.2 where $f(\langle p_1(A)\rangle \times \{l\} \times \{\alpha, \beta\}, E_l)$ is a subdigraph of Δ induced by $f(\langle p_1(A)\rangle \times \{l\} \times \{\alpha, \beta\})$.

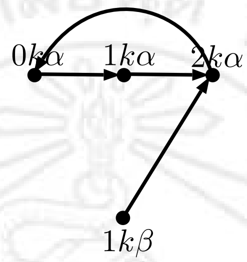

Figure 7.2: Subdigraph of Δ induced by $\{0k\alpha, 1k\alpha, 2k\alpha, 1k\beta\}$.

Similarly, we can observe that the digraph $f(\langle p_1(A)\rangle \times \{k\} \times \{\alpha, \beta\}, E_k)$ is a subdigraph of $(\langle p_1(A)\rangle \times \{l\} \times \{\alpha, \beta\}, E_l)$.

Moreover, we have

$$\Phi_{l\alpha} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \Phi_{k\alpha} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

which are endomorphisms on $\mathrm{Cay}(\mathbb{Z}_3, \{1\})$.

In addition, f satisfies the condition (3) in Theorem 7.1.1 as shown as follows:

$f(0 + 1^{-1}, l, \alpha) = f(2, l, \alpha) = (0, k, \alpha) = (1 + 2, k, \alpha) = (\Phi_{l\alpha}(0) + 1^{-1}, k, \alpha)$,

$f(1 + 1^{-1}, l, \alpha) = f(0, l, \alpha) = (1, k, \alpha) = (2 + 2, k, \alpha) = (\Phi_{l\alpha}(1) + 1^{-1}, k, \alpha)$,

$f(2 + 1^{-1}, l, \alpha) = f(1, l, \alpha) = (2, k, \alpha) = (0 + 2, k, \alpha) = (\Phi_{l\alpha}(2) + 1^{-1}, k, \alpha)$,

$f(0 + 1^{-1}, k, \alpha) = f(2, k, \alpha) = (1, l, \alpha) = (2 + 2, l, \alpha) = (\Phi_{k\alpha}(0) + 1^{-1}, l, \alpha)$,

$f(1 + 1^{-1}, k, \alpha) = f(0, k, \alpha) = (2, l, \alpha) = (0 + 2, l, \alpha) = (\Phi_{k\alpha}(1) + 1^{-1}, l, \alpha)$,

$f(2 + 1^{-1}, k, \alpha) = f(1, k, \alpha) = (0, l, \alpha) = (1 + 2, l, \alpha) = (\Phi_{k\alpha}(2) + 1^{-1}, l, \alpha)$.

Similarly, for each $t \in \{l, k\}$, we have $f(x + 1^{-1}, t, \beta) \in \{\Phi_{t\alpha}(x) + 1^{-1}\} \times \{u\} \times R$ for all $x \in \mathbb{Z}_3$ and for some $u \in \{l, k\}$.

The next proposition describes the relation between two mappings for studying an endomorphism of a Cayley digraph Δ of a rectangular group $G \times L \times R$ with respect to the connection set mentioned in the above theorem via an arc-preserving property. Before we show the result, we need to define the notation for convenience to use in the proof.

Let $f : \Delta \to \Delta$, $l \in L$ and $\alpha \in R$. We denote by $f_{Gl\alpha}$ the restriction function

$$f_{|_{G\times\{l\}\times\{\alpha\}}} : (G \times \{l\} \times \{\alpha\}, E_{l\alpha}) \to \Delta$$

where $E_{l\alpha}$ is an arc set of the strong subdigraph $(G \times \{l\} \times \{\alpha\}, E_{l\alpha})$ of Δ.

Proposition 7.1.3. *Let $A = K \times P \times T$ be a connection set of Δ. For each $l \in L$ and $\alpha \in T$, the function $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G,K))$ if and only if $p_1 \circ f_{Gl\alpha}$ is arc-preserving.*

Proof. Let $l \in L$ and $\alpha \in T$. Suppose that $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G,K))$. Let $g, h \in G$ be such that $((g,l,\alpha),(h,l,\alpha)) \in E(\Delta)$. Then there exists $(a,q,\lambda) \in A$ such that

$$(h,l,\alpha) = (g,l,\alpha)(a,q,\lambda) = (ga,l,\lambda),$$

that is, $h = ga$ where $a \in K$ which implies that $(g,h) \in E(\mathrm{Cay}(G,K))$. Since $\Phi_{l\alpha}$ is an endomorphism of $\mathrm{Cay}(G,K)$, we have $(\Phi_{l\alpha}(g), \Phi_{l\alpha}(h)) \in E(\mathrm{Cay}(G,K))$. We may assume that $\Phi_{l\alpha}(g) = x$ and $\Phi_{l\alpha}(h) = y$ for some $x, y \in G$. Thus

$$f_{Gl\alpha}(g,l,\alpha) = f(g,l,\alpha) = (x,t,\mu) \text{ and } f_{Gl\alpha}(h,l,\alpha) = f(h,l,\alpha) = (y,s,\eta)$$

for some $s, t \in L$ and $\mu, \eta \in R$. Hence

$$(p_1 \circ f_{Gl\alpha})(h,l,\alpha) = y = \Phi_{l\alpha}(h) = \Phi_{l\alpha}(g)k = xk = (p_1 \circ f_{Gl\alpha})(g,l,\alpha)k$$

where $k \in K$. Then $((p_1 \circ f_{Gl\alpha})(g,l,\alpha), (p_1 \circ f_{Gl\alpha})(h,l,\alpha)) \in E(\Delta)$. Therefore, $p_1 \circ f_{Gl\alpha}$ is arc-preserving.

Conversely, assume that $p_1 \circ f_{Gl\alpha}$ is arc-preserving. We want to show that $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G,K))$. Let $x, y \in G$ be such that $(x,y) \in E(\mathrm{Cay}(G,K))$. Thus $y = xa$ for some $a \in K$. Since $\alpha \in T$, there exists $u \in P$ in which $(a,u,\alpha) \in A$ because $A = K \times P \times T$. Hence $(y,l,\alpha) = (xa,l,\alpha) = (x,l,\alpha)(a,u,\alpha)$ which leads to $((x,l,\alpha),(y,l,\alpha)) \in E(\Delta)$. By our assumption, we obtain that

$$((p_1 \circ f_{Gl\alpha})(x,l,\alpha), (p_1 \circ f_{Gl\alpha})(y,l,\alpha)) \in E(\mathrm{Cay}(G,K)).$$

We take $f(x,l,\alpha) = (x',l',\alpha')$ and $f(y,l,\alpha) = (y',l',\alpha')$ for some $(x',l',\alpha'), (y',l',\alpha') \in G \times L \times R$. Hence $\Phi_{l\alpha}(x) = x'$ and $\Phi_{l\alpha}(y) = y'$ which implies that

$$\begin{aligned}(\Phi_{l\alpha}(x), \Phi_{l\alpha}(y)) &= (x',y') \\ &= ((p_1 \circ f_{Gl\alpha})(x,l,\alpha), (p_1 \circ f_{Gl\alpha})(y,l,\alpha)) \in E(\mathrm{Cay}(G,K)).\end{aligned}$$

Consequently, $\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G,K))$. □

7.2 Endomorphisms on Cayley Digraphs of Left Groups

From the fact that a left group $G \times L$ is considered to be the special case of a rectangular group $G \times L \times R$, we are also attentive to characterize endomorphisms on a Cayley digraph Γ of a left group $G \times L$ with an arbitrary connection set A.

Before we show the characterization of endomorphisms on Cayley digraphs of left groups, we will define the following notation.

Let $G/\langle p_1(A)\rangle = \{g_1\langle p_1(A)\rangle, g_2\langle p_1(A)\rangle, \ldots, g_k\langle p_1(A)\rangle\}$ such that $g_i \in G$ for all $i \in I = \{1, 2, \ldots, k\}$ where $k \in \mathbb{N}$. Let $f : \Gamma \to \Gamma$ be a function from Γ into Γ itself and $l \in L$. We denote the restriction function $f|_{g_i\langle p_1(A)\rangle \times \{l\}} : (g_i\langle p_1(A)\rangle \times \{l\}, E_{il}) \to \Gamma$ by f_{il} in which $(g_i\langle p_1(A)\rangle \times \{l\}, E_{il})$ is the strong subdigraph of Γ.

Theorem 7.2.1. *Let Γ be a Cayley digraph* $\mathrm{Cay}(G \times L, A)$ *of a left group $G \times L$ with a connection set A and $f : \Gamma \to \Gamma$. The following statements are equivalent:*

(1). $f \in \mathrm{End}(\Gamma)$;

(2). f_{il} *is arc-preserving for all $l \in L$ and $i \in I$;*

(3). *for each $(x, l) \in G \times L$ and $a \in p_1(A)$, $f(xa, l) = (p_1(f(x, l))b, p_2(f(x, l)))$ for some $b \in p_1(A)$.*

Proof. Let A be a connection set of Γ and $f : \Gamma \to \Gamma$.

(1)$\Rightarrow$(2): Suppose that $f \in \mathrm{End}(\Gamma)$. Let $l \in L$ and $i \in I$. We will prove that f_{il} is arc-preserving. Let $((x, l), (y, l)) \in E(g_i\langle p_1(A)\rangle \times \{l\}, E_{il})$. Then $((x, l), (y, l)) \in E(\Gamma)$. We have $(f_{il}(x, l), f_{il}(y, l)) = (f(x, l), f(y, l)) \in E(\Gamma)$ since $f \in \mathrm{End}(\Gamma)$. Therefore, f_{il} is arc-preserving, as required.

(2)$\Rightarrow$(3): Assume that (2) is true. Let $(x, l) \in G \times L$ and $a \in p_1(A)$. We obtain that $(x, l) \in g_i\langle p_1(A)\rangle \times \{l\}$ for some $i \in I$. Thus there exists $l' \in p_2(A)$ such that $(a, l') \in A$. Consider $(xa, l) = (x, l)(a, l')$, we have $((x, l), (xa, l)) \in E(g_i\langle p_1(A)\rangle \times \{l\}, E_{il}) \subseteq E(\Gamma)$. Since f_{il} is arc-preserving, we get that $(f(x, l), f(xa, l)) = (f_{il}(x, l), f_{il}(xa, l)) \in E(\Gamma)$. Suppose that $f(x, l) = (y, l_1)$ for some $(y, l_1) \in G \times L$. Then there exists $(b, l_2) \in A$ such that

$$f(xa, l) = f(x, l)(b, l_2) = (y, l_1)(b, l_2) = (yb, l_1) = (p_1(f(x, l))b, p_2(f(x, l)))$$

where $b \in p_1(A)$.

(3)$\Rightarrow$(1): Suppose that the statement (3) holds. We will show that $f \in \mathrm{End}(\Gamma)$. Let $((x, l_1), (y, l_2)) \in E(\Gamma)$. Thus $(y, l_2) = (x, l_1)(a, l_3) = (xa, l_1)$ for some $(a, l_3) \in A$. Hence $y = xa$ and $l_1 = l_2$. Assume that $f(x, l_1) = (u, l_4)$ for some $(u, l_4) \in G \times L$. By our supposition, we obtain that

$$f(y,l_2) = f(xa,l_1) = (p_1(f(x,l_1))b, p_2(f(x,l_1))) = (ub,l_4)$$

for some $b_1 \in p_1(A)$. Since $b \in p_1(A)$, there exists $l_5 \in p_2(A)$ such that $(b,l_5) \in A$. We consequently obtain that

$$f(y,l_2) = (ub,l_4) = (u,l_4)(b,l_5) = f(x,l_1)(b,l_5),$$

that is, $(f(x,l_1), f(y,l_2)) \in E(\Gamma)$. Therefore, $f \in \mathrm{End}(\Gamma)$. $\square$

The following example is presented for guaranteeing the properties of endomorphisms on Cayley digraphs of left groups with arbitrary connection sets shown in the above theorem.

Example 7.2.2. Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_6 \times \{l,k\}, A)$ where $A = \{(2,l)\}$.

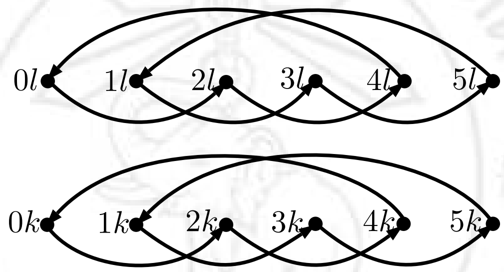

Figure 7.3: $\mathrm{Cay}(\mathbb{Z}_6 \times \{l,k\}, \{(2,l)\})$.

We obtain that

$$f = \begin{pmatrix} 0l & 1l & 2l & 3l & 4l & 5l & 0k & 1k & 2k & 3k & 4k & 5k \\ 1k & 2l & 3k & 4l & 5k & 0l & 3l & 5l & 5l & 1l & 1l & 3l \end{pmatrix} \in \mathrm{End}(\Gamma).$$

Since $\langle p_1(A)\rangle = \langle\{2\}\rangle = \{0,2,4\}$, if we let $g_1 = 0$ and $g_2 = 1$, then we obtain that

$$(g_1 + \langle p_1(A)\rangle) \times \{l\} = \{(0,l),(2,l),(4,l)\};$$
$$(g_1 + \langle p_1(A)\rangle) \times \{k\} = \{(0,k),(2,k),(4,k)\};$$
$$(g_2 + \langle p_1(A)\rangle) \times \{l\} = \{(1,l),(3,l),(5,l)\} \text{ and}$$
$$(g_2 + \langle p_1(A)\rangle) \times \{k\} = \{(1,k),(3,k),(5,k)\}.$$

We can conclude that

$$f_{1l} = \begin{pmatrix} 0l & 2l & 4l \\ 1k & 3k & 5k \end{pmatrix} \quad \text{and} \quad f_{1k} = \begin{pmatrix} 0k & 2k & 4k \\ 3l & 5l & 1l \end{pmatrix};$$

$$f_{2l} = \begin{pmatrix} 1l & 3l & 5l \\ 2l & 4l & 0l \end{pmatrix} \quad \text{and} \quad f_{2k} = \begin{pmatrix} 1k & 3k & 5k \\ 5l & 1l & 3l \end{pmatrix}$$

and they are arc-preserving.

The following computation shows that the endomorphism f defined as above satisfies the third condition referred in Theorem 7.2.1.

$f(0+2,l) = f(2,l) = (3,k) = (1+2,k) = (p_1(f(0,l))+2, p_2(f(0,l)))$,
$f(1+2,l) = f(3,l) = (4,l) = (2+2,l) = (p_1(f(1,l))+2, p_2(f(1,l)))$,
$f(2+2,l) = f(4,l) = (5,k) = (3+2,k) = (p_1(f(2,l))+2, p_2(f(2,l)))$,
$f(3+2,l) = f(5,l) = (0,l) = (4+2,l) = (p_1(f(3,l))+2, p_2(f(3,l)))$,
$f(4+2,l) = f(0,l) = (1,k) = (5+2,k) = (p_1(f(4,l))+2, p_2(f(4,l)))$,
$f(5+2,l) = f(1,l) = (2,l) = (0+2,l) = (p_1(f(5,l))+2, p_2(f(5,l)))$.

Similarly, we obtain that $f(x+2,k) = (p_1(f(x,k))+2, p_2(f(x,k)))$ for all $x \in \mathbb{Z}_6$.

7.3 Endomorphisms on Cayley Digraphs of Right Groups

As the fact that a Cayley digraph Δ of a rectangular group $G \times L \times R$ with a connection set A is a disjoint union of $|L|$ isomorphic subdigraphs which each subdigraph is isomorphic to a Cayley digraph Λ of a right group $G \times R$ with a connection set $\overline{A}$ defined as follows:

$$\overline{A} = \{(a,\alpha) \in G \times R : (a,l,\alpha) \in A \text{ for some } l \in L\},$$

we are also interested in the structure of endomorphisms on a Cayley digraph Λ of a right group $G \times R$ where the connection set A is in the form of the cartesian product of sets.

Before we present some results of endomorphisms on Cayley digraphs of right groups, we first define the gainful function which is used in this part.

Let $f : \Lambda \to \Lambda$ be a function from Λ into Λ itself. For each $\alpha \in R$ and for all $a \in G$, we define $\varphi_\alpha : \mathrm{Cay}(G, p_1(A)) \to \mathrm{Cay}(G, p_1(A))$ by

$$\varphi_\alpha(a) = b \text{ if and only if there exists } \beta \in R \text{ such that } f(a,\alpha) = (b,\beta).$$

It is not hard to examine that φ_α is well-defined. The following theorem shows some results of endomorphisms on Cayley digraphs of right groups with some connection sets.

Theorem 7.3.1. *Let $A = K \times T$ be a connection set of Λ. If $f : \Lambda \to \Lambda$, then $f \in \mathrm{End}(\Lambda)$ if and only if the following statements hold:*

(1). $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G,K))$ *for all* $\alpha \in T$;

(2). *for each* $g \in K$ *and* $a \in G$, *there exists* $g_a \in K$ *such that*

$$f(ag^{-1},\theta) \in \begin{cases} \{\varphi_\lambda(a)g_a^{-1}\} \times T & \text{if } \theta \in T, \\ \{\varphi_\lambda(a)g_a^{-1}\} \times R & \text{if } \theta \in R \setminus T \end{cases}$$

for all $\lambda \in T$.

Proof. Let $A = K \times T$ be a connection set of Λ and $f : \Lambda \to \Lambda$.
($\Rightarrow$) Suppose that $f \in \mathrm{End}(\Lambda)$.

Let $\alpha \in T$. We first show that $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G,K))$. Assume that $(x,y) \in E(\mathrm{Cay}(G,K))$. Thus $y = xa$ for some $a \in K$. We will let $\varphi_\alpha(x) = u$ and $\varphi_\alpha(y) = v$ for some $u, v \in G$. Then there exist $\gamma, \eta \in R$ such that $f(x,\alpha) = (u,\gamma)$ and $f(y,\alpha) = (v,\eta)$, respectively. Since $f \in \mathrm{End}(\Lambda)$ and

$$((x,\alpha),(y,\alpha)) = ((x,\alpha),(xa,\alpha)) = ((x,\alpha),(x,\alpha)(a,\lambda)) \in E(\Lambda)$$

for some $\lambda \in T$, we obtain that $(f(x,\alpha), f(y,\alpha)) \in E(\Lambda)$. Thus $f(y,\alpha) = f(x,\alpha)(w,\rho)$ for some $(w,\rho) \in A$ which implies that

$$(v,\eta) = f(y,\alpha) = f(x,\alpha)(w,\rho) = (u,\gamma)(w,\rho) = (uw,\rho).$$

So we can conclude that $v = uw$, that is, $\varphi_\alpha(y) = v = uw = \varphi_\alpha(x)w$ where $w \in K$. Therefore, $(\varphi_\alpha(x), \varphi_\alpha(y)) \in E(\mathrm{Cay}(G,K))$ that means $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G,K))$.

Let $\lambda \in T$ and $\theta \in R$. For each $g \in K$ and $a \in G$, let us consider $(ag^{-1},\theta) \in G \times R$. Since $(a,\lambda) = (ag^{-1},\theta)(g,\lambda)$ where $(g,\lambda) \in K \times T = A$, we can directly obtain that $((ag^{-1},\theta),(a,\lambda)) \in E(\Lambda)$. From $f \in \mathrm{End}(\Lambda)$, we have $(f(ag^{-1},\theta), f(a,\lambda)) \in E(\Lambda)$, that is, $f(a,\lambda) = f(ag^{-1},\theta)(g_a,\beta)$ for some $(g_a,\beta) \in A$. We may assume that $f(ag^{-1},\theta) = (x,\mu)$ for some $x \in G$ and $\mu \in R$. Then

$$f(a,\lambda) = f(ag^{-1},\theta)(g_a,\beta) = (x,\mu)(g_a,\beta) = (xg_a,\beta)$$

which leads to $\varphi_\lambda(a) = xg_a$. Therefore,

$$f(ag^{-1},\theta) = (x,\mu) = (xg_ag_a^{-1},\mu) = (\varphi_\lambda(a)g_a^{-1},\mu).$$

If $\theta \in T$, then $(g,\theta) \in K \times T = A$. Since $(ag^{-1},\theta) = (ag^{-2},\theta)(g,\theta)$, we get that $((ag^{-2},\theta),(ag^{-1},\theta)) \in E(\Lambda)$ which implies that $(f(ag^{-2},\theta), f(ag^{-1},\theta)) \in E(\Lambda)$. Suppose that $f(ag^{-2},\theta) = (c,\varepsilon)$ for some $(c,\varepsilon) \in G \times R$. Hence

$$f(ag^{-1},\theta) = f(ag^{-2},\theta)(m,\delta) = (c,\varepsilon)(m,\delta) = (cm,\delta)$$

for some $(m,\delta) \in A$. We have $(\varphi_\lambda(a)g_a^{-1}, \mu) = (cm, \delta)$ which leads to $\mu = \delta \in T$. Consequently, we can totally conclude that

$$f(ag^{-1}, \theta) \in \begin{cases} \{\varphi_\lambda(a)g_a^{-1}\} \times T & \text{if } \theta \in T, \\ \{\varphi_\lambda(a)g_a^{-1}\} \times R & \text{if } \theta \in R \setminus T. \end{cases}$$

($\Leftarrow$) Assume that the conditions hold. We will prove that $f \in \mathrm{End}(\Lambda)$. Let $((x,\rho),(y,\lambda)) \in E(\Lambda)$. Thus $(y,\lambda) = (x,\rho)(k,\theta) = (xk,\theta)$ for some $(k,\theta) \in A$ and then $y = xk$ and $\lambda = \theta \in T$. Since $\varphi_\lambda \in \mathrm{End}(\mathrm{Cay}(G,K))$ and $(x,y) = (x,xk) \in E(\mathrm{Cay}(G,K))$ because $k \in K$, we gain that $(\varphi_\lambda(x), \varphi_\lambda(y)) \in E(\mathrm{Cay}(G,K))$. Hence $\varphi_\lambda(y) = \varphi_\lambda(x)s$ for some $s \in K$. By our assumption, we obtain that there exist $t \in K$ and $\mu \in R$ in which

$$f(x,\rho) = f(xkk^{-1}, \rho) = (\varphi_\lambda(xk)t^{-1}, \mu) = (\varphi_\lambda(y)t^{-1}, \mu) = (\varphi_\lambda(x)st^{-1}, \mu).$$

Since $\lambda \in T$, we have by our assumption that there exists $u \in K$ such that

$$f(y,\lambda) = f(yzz^{-1}, \lambda) = (\varphi_\lambda(yz)u^{-1}, \xi)$$

for some $z \in K$ and $\xi \in T$. Consider the definition of φ_λ, we have $f(y,\lambda) = (\varphi_\lambda(y), \beta)$ for some $\beta \in R$. We get that $\beta = \xi \in T$. Hence there exists $(t,\beta) \in K \times T = A$ in which

$$f(y,\lambda) = (\varphi_\lambda(y), \beta) = (\varphi_\lambda(x)s, \beta) = (\varphi_\lambda(x)st^{-1}, \mu)(t,\beta) = f(x,\rho)(t,\beta).$$

Therefore, $(f(x,\rho), f(y,\lambda)) \in E(\Lambda)$, that is, $f \in \mathrm{End}(\Lambda)$, as desired. $\square$

We now present an example of an endomorphism on a Cayley digraph of a right group with a connection set mentioned in Theorem 7.3.1.

Example 7.3.2. Let $\Lambda = \mathrm{Cay}(\mathbb{Z}_6 \times \{\alpha, \beta\}, A)$ where $A = \{2\} \times \{\alpha\}$.

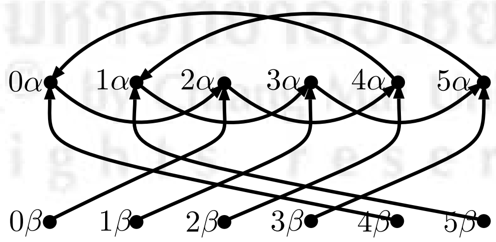

Figure 7.4: $\mathrm{Cay}(\mathbb{Z}_6 \times \{\alpha, \beta\}, \{(2,\alpha)\})$.

We obtain that

$$f = \begin{pmatrix} 0\alpha & 1\alpha & 2\alpha & 3\alpha & 4\alpha & 5\alpha & 0\beta & 1\beta & 2\beta & 3\beta & 4\beta & 5\beta \\ 5\alpha & 3\alpha & 1\alpha & 5\alpha & 3\alpha & 1\alpha & 5\beta & 3\alpha & 1\alpha & 5\alpha & 3\beta & 1\alpha \end{pmatrix} \in \mathrm{End}(\Lambda).$$

Moreover, we have

$$\varphi_\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 5 & 3 & 1 \end{pmatrix} \in \mathrm{End}(\mathrm{Cay}(\mathbb{Z}_6, \{2\})).$$

In addition, f satisfies the condition (2) in Theorem 7.3.1 as shown as follows:

$$\begin{aligned}
f(0 + 2^{-1}, \alpha) &= f(4, \alpha) = (3, \alpha) = (5 + 4, \alpha) = (\varphi_\alpha(0) + 2^{-1}, \alpha), \\
f(1 + 2^{-1}, \alpha) &= f(5, \alpha) = (1, \alpha) = (3 + 4, \alpha) = (\varphi_\alpha(1) + 2^{-1}, \alpha), \\
f(2 + 2^{-1}, \alpha) &= f(0, \alpha) = (5, \alpha) = (1 + 4, \alpha) = (\varphi_\alpha(2) + 2^{-1}, \alpha), \\
f(3 + 2^{-1}, \alpha) &= f(1, \alpha) = (3, \alpha) = (5 + 4, \alpha) = (\varphi_\alpha(3) + 2^{-1}, \alpha), \\
f(4 + 2^{-1}, \alpha) &= f(2, \alpha) = (1, \alpha) = (3 + 4, \alpha) = (\varphi_\alpha(4) + 2^{-1}, \alpha), \\
f(5 + 2^{-1}, \alpha) &= f(3, \alpha) = (5, \alpha) = (1 + 4, \alpha) = (\varphi_\alpha(5) + 2^{-1}, \alpha).
\end{aligned}$$

Similarly, we have $f(x + 2^{-1}, \beta) \in \{\varphi_\alpha(x) + 2^{-1}\} \times R$ for all $x \in \mathbb{Z}_6$.

The next result gives the necessary and sufficient conditions for endomorphisms on Cayley digraphs of right groups which connection sets are in the form $\{g\} \times R$, the special case of the connection set in Theorem 7.3.1.

Corollary 7.3.3. *Let $A = \{g\} \times R$ be a connection set of Λ where $g \in G$. If $f : \Lambda \to \Lambda$, then $f \in \mathrm{End}(\Lambda)$ if and only if the following statements hold:*

(1). *$\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G, \{g\}))$ for all $\alpha \in R$;*

(2). *$\varphi_\beta = \varphi_\gamma$ for all $\beta, \gamma \in R$.*

Proof. Let $A = \{g\} \times R$ be a connection set of Λ for some $g \in G$.

Suppose that $f \in \mathrm{End}(\Lambda)$. By Theorem 7.3.1, we have $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G, \{g\}))$ for all $\alpha \in R$. Next, we will prove the statement (2). Let $\beta, \gamma \in R$ and $x \in G$. Suppose that $\varphi_\beta(x) = s$ and $\varphi_\gamma(x) = t$ for some $s, t \in G$. Then $f(x, \beta) = (s, \lambda)$ and $f(x, \gamma) = (t, \mu)$ for some $\lambda, \mu \in R$. Since $(xg, \gamma) = (x, \beta)(g, \gamma)$ and $(xg, \gamma) = (x, \gamma)(g, \gamma)$ where $(g, \gamma) \in A$, we have $((x, \beta), (xg, \gamma)), ((x, \gamma), (xg, \gamma)) \in E(\Lambda)$. Hence $(f(x, \beta), f(xg, \gamma)), (f(x, \gamma), f(xg, \gamma)) \in E(\Lambda)$ since $f \in \mathrm{End}(\Lambda)$. Thus

$$f(xg, \gamma) = f(x, \beta)(g, \delta) \text{ and } f(xg, \gamma) = f(x, \gamma)(g, \eta)$$

for some $\delta, \eta \in R$. We conclude that $f(x, \beta)(g, \delta) = f(x, \gamma)(g, \eta)$ which implies that

$$(sg, \delta) = (s, \lambda)(g, \delta) = f(x, \beta)(g, \delta) = f(x, \gamma)(g, \eta) = (t, \mu)(g, \eta) = (tg, \eta).$$

Hence $sg = tg$ and then $s = t$ by the cancellation law of a group G. Therefore,

$$\varphi_\beta(x) = s = t = \varphi_\gamma(x), \text{ that is, } \varphi_\beta = \varphi_\gamma, \text{ as required.}$$

Conversely, assume that (1) and (2) are true. We want to show that $f \in \mathrm{End}(\Lambda)$. Let $(x,\alpha),(y,\beta) \in E(\Lambda)$. Then $(y,\beta) = (x,\alpha)(g,\lambda) = (xg,\lambda)$ for some $\lambda \in R$ which implies that $y = xg$. Thus $(x,y) \in E(\mathrm{Cay}(G,\{g\}))$. By our assumption, we obtain that $\varphi_\alpha, \varphi_\beta \in \mathrm{End}(\mathrm{Cay}(G,\{g\}))$ and $\varphi_\alpha = \varphi_\beta$. We have $(\varphi_\alpha(x), \varphi_\alpha(y)) \in E(\mathrm{Cay}(G,\{g\}))$, that is, $\varphi_\beta(y) = \varphi_\alpha(y) = \varphi_\alpha(x)g$. Hence there exists $\theta \in R$ such that $f(y,\beta) = (\varphi_\alpha(x)g,\theta)$. Suppose that $\varphi_\alpha(x) = u$ for some $u \in G$ that means there exists $\mu \in R$ in which $f(x,\alpha) = (u,\mu)$. Therefore, $f(y,\beta) = (\varphi_\alpha(x)g,\theta) = (ug,\theta) = (u,\mu)(g,\theta) = f(x,\alpha)(g,\theta)$ where $(g,\theta) \in \{g\} \times R = A$. Hence $(f(x,\alpha), f(y,\beta)) \in E(\Lambda)$. Consequently, $f \in \mathrm{End}(\Lambda)$. $\square$

Furthermore, the number of endomorphisms on a Cayley digraph of a right group with the connection set $\{g\} \times R$ is obtained in the following proposition.

Proposition 7.3.4. *Let G be a group of order n and R a right zero semigroup of order m where $m,n \in \mathbb{N}$. Let $A = \{g\} \times R$ be a connection set of Λ where $g \in G$.*
If $|\mathrm{End}(\mathrm{Cay}(G,\{g\}))| = d$ for some $d \in \mathbb{N}$, then $|\mathrm{End}(\Lambda)| = d \cdot m^{mn}$.

Proof. Let $A = \{g\} \times R$ be a connection set of the Cayley digraph Λ where $g \in G$. Suppose that $|\mathrm{End}(\mathrm{Cay}(G,\{g\}))| = d$ for some $d \in \mathbb{N}$. In order to construct an endomorphism f of Λ, let $\phi \in \mathrm{End}(\mathrm{Cay}(G,\{g\}))$ be fixed. For each $(x,\alpha) \in G \times R$, we define $f : \Lambda \to \Lambda$ by

$$f(x,\alpha) = (\phi(x),\beta) \text{ for some } \beta \in R.$$

Then f is an endomorphism of Λ followed from Corollary 7.3.3. It can be easily seen that β is arbitrary, this means that it does not matter when we choose whatever $\beta \in R$, the function f is always an endomorphism of Λ. So we can conclude that for each $\phi \in \mathrm{End}(\mathrm{Cay}(G,\{g\}))$ and for each $(x,\alpha) \in G \times R$, we have m ways to construct endomorphisms of Λ. On the other hand, if we pick $f \in \mathrm{End}(\Lambda)$, we can obtain by Corollary 7.3.3 that f must be one of those functions that we defined above. Consequently, we can conclude that $|\mathrm{End}(\Lambda)| = |\mathrm{End}(\mathrm{Cay}(G,\{g\}))||R|^{|G\times R|} = d \cdot m^{mn}$, as required. $\square$