CHAPTER 8

Conclusions

In summary of the thesis, we have provided the backgrounds for the research and some preliminaries together with useful notations in Chapter 1 and Chapter 2, respectively. All of preliminaries have been divided into two parts consisting of semigroup theory and graph theory. Some related examples are also illustrated in this chapter.

For our main results, we have studied the characterizations of Cayley digraphs of rectangular groups. We have found that the Cayley digraph of a rectangular group is the disjoint union of isomorphic strong subdigraphs which each subdigraph is isomorphic to the Cayley digraph of a right group with an according connection set.

Moreover, we have provided some results of domination parameters consisting of the domination number and total domination number of Cayley digraphs of rectangular groups with their connection sets. In fact, since left groups and right groups are rectangular groups, we have also considered such semigroups in this thesis.

The following theorem shows a lower bound and an upper bound of the domination number in Cayley digraphs of left groups with their corresponding connection sets.

Theorem 8.1. *Let $G \times L$ be a left group and A a connection set of Γ such that the identity of G lies in $p_1(A)$. If H is a subgroup of G with a maximum cardinality and contained in $p_1(A)$, then $\frac{|G|}{|p_1(A)|}|L| \leq \gamma(\Gamma) \leq [G : H]|L|$ where $[G : H]$ is the index of H in G.*

We continue with the following results which describe the domination number of Cayley digraphs of right groups with arbitrary connection sets.

Theorem 8.2. *Let $G \times R$ be a right group and A a connection set of Λ. If $|p_2(A)| \neq |R|$, then $\gamma(\Lambda) = (|R| - |p_2(A)|) \times |G|$.*

Theorem 8.3. *Let $G \times R$ be a right group and A a connection set of Λ. If $|p_2(A)| = |R|$, then $\frac{|G||R|}{|A|+1} \leq \gamma(\Lambda) \leq |G|$.*

Furthermore, the sharpness of those bounds of $\gamma(\Lambda)$ stated in Theorem 8.3 has been completely proved.

Some results of the total domination number have been presented. The following theorem gives the necessary and sufficient conditions for an existence of total dominating sets in Cayley digraphs of left groups.

Theorem 8.4. *Let $G \times L$ be a left group and A a connection set of Γ. Then the total dominating set of Γ exists if and only if $A \neq \emptyset$.*

For the necessary and sufficient conditions for the existence of total dominating sets in Cayley digraphs of right groups, we will present in the following theorem.

Theorem 8.5. *Let $G \times R$ be a right group and A a connection set of Λ. Then the total dominating set of Λ exists if and only if $p_2(A) = R$.*

In addition, a lower bound and an upper bound of the total domination number of Cayley digraphs of right groups are shown in the following theorem.

Theorem 8.6. *Let $G \times R$ be a right group and A a connection set of Λ such that $p_2(A) = R$. Then $\frac{|G||R|}{|A|} \leq \gamma_t(\Lambda) \leq |G|$.*

Besides those domination parameters, we have also proposed some results about the independence parameters consisting of the independence number, weakly independence number, dipath independence number, and weakly dipath independence number on Cayley digraphs of rectangular groups.

Some results of the parameter α, the independence number, of a digraph Δ are presented as follows.

Theorem 8.7. *If I is an $\alpha-$set of Δ, then $I \cap (G \times \{\ell\} \times R)$ is an $\alpha-$set of the digraph $(G \times \{\ell\} \times R, E_\ell)$ for all $\ell \in L$.*

Theorem 8.8. *Let $\overline{A} = \{(a, \lambda) \in G \times R : (a, l, \lambda) \in A$ for some $l \in L\}$ where A is a connection set of Δ. If T is an $\alpha-$set of $\mathrm{Cay}(G \times R, \overline{A})$, then $\bigcup\limits_{(t,\lambda) \in T} (\{t\} \times L \times \{\lambda\})$ is an $\alpha-$set of Δ.*

Moreover, by considering the above two theorems, we have similarly obtained the results for the weakly independence number, dipath independence number, and weakly dipath independence number on Cayley digraphs of rectangular groups.

For the independence number of Cayley digraphs of left groups and right groups, we show the results as follows.

Theorem 8.9. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. Then* $\frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lceil \frac{|\langle p_1(A)\rangle|}{2|p_1(A)|+1} \right\rceil \leq \alpha(\Gamma) \leq \frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor$.

Theorem 8.10. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A in which $p_2(A) \neq R$. If $|R| \geq 2|p_2(A)|$, then $\alpha(\Lambda) = (|R| - |p_2(A)|)|G|$.*

Theorem 8.11. *Let Λ denote a Cayley digraph of a right group $G \times R$ with a connection set A in which $p_2(A) = R$. Then* $|R| \leq \alpha(\Lambda) \leq \left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor \frac{|G||R|}{|\langle p_1(A)\rangle|}$.

In addition, we have proved that those bounds given in Theorem 8.9 and Theorem 8.11 are sharp.

Some facts about the properties of the weakly independence number, the parameter α_w, are shown as follows.

Theorem 8.12. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. If $p_1(A) = [p_1(A)]^{-1}$ where $[p_1(A)]^{-1} = \{x^{-1} : x \in p_1(A)\}$, then $\alpha_w(\Gamma) = \alpha(\Gamma)$.*

Theorem 8.13. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. Then* $\frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lceil \frac{|\langle p_1(A)\rangle|}{2|p_1(A)|+1} \right\rceil \leq \alpha_w(\Gamma) \leq \frac{|G||L|}{|\langle p_1(A^\sharp)\rangle|}\left\lfloor \frac{|\langle p_1(A^\sharp)\rangle|}{2} \right\rfloor$.

Theorem 8.14. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. Then $\alpha_w(\Lambda) = |G||R|$ if and only if $p_1(A) \cap [p_1(A)]^{-1} = \emptyset$.*

For the parameter α_p, the dipath independence number, we have obtained the exact values of the dipath independence number for Cayley digraphs of left groups and right groups, respectively.

Theorem 8.15. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. Then* $\alpha_p(\Gamma) = \frac{|G||L|}{|\langle p_1(A)\rangle|}$.

Theorem 8.16. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. If $p_2(A) \neq R$, then $\alpha_p(\Lambda) = |G|(|R| - |p_2(A)|)$.*

Theorem 8.17. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. If $p_2(A) = R$, then $\alpha_p(\Lambda) = \frac{|G|}{|\langle p_1(A)\rangle|}$.*

We have also considered the weakly dipath independence number, the parameter α_{wp}, of Cayley digraphs of left groups and right groups. For Cayley digraphs of left groups, we have concluded that the results of the weakly dipath independence number and the dipath independence number are same. For the weakly dipath independence number of Cayley digraphs of right groups, the result is shown as follows.

Theorem 8.18. *Let Λ be a Cayley digraph of a right group $G \times R$ with a connection set A. Then $\alpha_{wp}(\Lambda) = \frac{|G|}{|\langle p_1(A)\rangle|} + |G|(|R| - |p_2(A)|)$.*

Furthermore, the independent domination parameters; i, i_w, i_p, and i_{wp}; on Cayley digraphs of rectangular groups, left groups, and right groups have been presented. For those independent domination parameters on Cayley digraphs of rectangular groups, we have also obtained the results which are similar to Theorem 8.7 and Theorem 8.8. In addition, the sufficient condition for the existence of the independent domination number $i(\Gamma)$ of a Cayley digraph Γ of a left group has been described as shown in the following theorem.

Theorem 8.19. *Let Γ be a Cayley digraph of a left group $G \times L$ with a connection set A. If $p_1(A) = [p_1(A)]^{-1}$, then $i(\Gamma)$ exists.*

We have also provided a lower bound and an upper bound of $i(\Gamma)$ under the above sufficient condition as follows.

Theorem 8.20. *Let A be a connection set of Γ where the identity $e \notin p_1(A)$. If $p_1(A) = [p_1(A)]^{-1}$, then $\frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lceil \frac{|\langle p_1(A)\rangle|}{|p_1(A)|+1} \right\rceil \leq i(\Gamma) \leq \frac{|G||L|}{|\langle p_1(A)\rangle|}\left\lfloor \frac{|\langle p_1(A)\rangle|}{2} \right\rfloor$.*

The following theorem presents an exact value of the independent domination number of a Cayley digraph Λ of a right group under the condition that $p_2(A) \neq R$.

Theorem 8.21. *If $p_2(A) \neq R$, then $i(\Lambda) = |G|(|R| - |p_2(A)|)$.*

Some results for the dipath independent domination number of Cayley digraphs of rectangular groups with their connection sets have been introduced as follows.

Theorem 8.22. *Let Δ be a Cayley digraph of a rectangular group $G \times L \times R$ with a connection set A. If $p_3(A) \neq R$, then $i_p(\Delta) = |G||L|(|R| - |p_3(A)|)$.*

We have given the necessary and sufficient conditions for the existence of a dipath independent domination number of the Cayley digraph Δ of a rectangular group under the condition that $p_3(A) = R$ as shown in the following theorem.

Theorem 8.23. *If $p_3(A) = R$, then the following statements are equivalent:*
(1). *$i_p(\Delta)$ exists;* (2). *$\overline{A} = \langle p_1(A) \rangle \times R$;* (3). *$i_p(\Delta) = \frac{|G||L|}{|\langle p_1(A) \rangle|}$.*

For the weakly dipath independent domination number, the parameter i_{wp}, of the Cayley digraph Δ of a rectangular group, we have concluded that $i_{wp}(\Delta)$ and $i_p(\Delta)$ coincide.

The last part of our main results has been provided to present some results for endomorphisms on Cayley digraphs of rectangular groups, left groups, and right groups. We have given the characterization of endomorphisms on Cayley digraphs of rectangular groups where the connection sets are in the form of a direct product of sets as shown in the following theorem.

Theorem 8.24. *Let $A = K \times P \times T$ be a connection set of Δ and $l \in L$. If $f : \Delta \to \Delta$, then $f \in \mathrm{End}(\Delta)$ if and only if the following statements hold:*

(1). *$f(b\langle K \rangle \times \{l\} \times R, E_l)$ is a subdigraph of $(c\langle K \rangle \times \{t\} \times R, E_t)$ for some $t \in L$, $c \in G$ and for all $b \in G$;*

(2). *$\Phi_{l\alpha} \in \mathrm{End}(\mathrm{Cay}(G, K))$ for all $\alpha \in T$;*

(3). *for each $g \in K$ and $a \in G$, there exists $g_a \in K$ such that*

$$f(ag^{-1}, l, \theta) \in \begin{cases} \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times T & \text{if } \theta \in T, \\ \{\Phi_{l\lambda}(a)g_a^{-1}\} \times \{u\} \times R & \text{if } \theta \in R \setminus T \end{cases}$$

for all $\lambda \in T$ and for some $u \in L$.

Moreover, the endomorphisms on Cayley digraphs of left groups with arbitrary connection sets have been characterized.

Theorem 8.25. *Let* Γ *be a Cayley digraph* $\mathrm{Cay}(G \times L, A)$ *of a left group* $G \times L$ *with a connection set* A *and* $f : \Gamma \to \Gamma$. *The following statements are equivalent:*

(1). $f \in \mathrm{End}(\Gamma)$*;*

(2). f_{il} *is arc-preserving for all* $l \in L$ *and* $i \in I$*;*

(3). *for each* $(x, l) \in G \times L$ *and* $a \in p_1(A)$, $f(xa, l) = (p_1(f(x, l))b, p_2(f(x, l)))$ *for some* $b \in p_1(A)$.

The last theorem shows the result of endomorphisms on Cayley digraphs of right groups with the connection sets given in the form of a direct product of sets.

Theorem 8.26. *Let* $A = K \times T$ *be a connection set of* Λ. *If* $f : \Lambda \to \Lambda$, *then* $f \in \mathrm{End}(\Lambda)$ *if and only if the following statements hold:*

(1). $\varphi_\alpha \in \mathrm{End}(\mathrm{Cay}(G, K))$ *for all* $\alpha \in T$*;*

(2). *for each* $g \in K$ *and* $a \in G$, *there exists* $g_a \in K$ *such that*

$$f(ag^{-1}, \theta) \in \begin{cases} \{\varphi_\lambda(a)g_a^{-1}\} \times T & \text{if } \theta \in T, \\ \{\varphi_\lambda(a)g_a^{-1}\} \times R & \text{if } \theta \in R \setminus T \end{cases}$$

for all $\lambda \in T$.

Furthermore, some related examples satisfying such theorems have been certainly illustrated to guarantee the properties of endomorphisms on those Cayley digraphs of rectangular groups, left groups, and right groups with their corresponding connection sets, respectively.