CHAPTER 1

Introduction

Let S be a semigroup and $A \subseteq S$. The *Cayley digraph* $\mathrm{Cay}(S, A)$ of a semigroup S relative to A is defined as the digraph with vertex set S and edge set $E(\mathrm{Cay}(S, A))$ consisting of those ordered pairs (x, y) such that $y = xa$ for some $a \in A$. In the first instance, Arthur Cayley introduced the Cayley graphs of finite groups in 1878. The Cayley graphs of semigroups were introduced by Zelinka in 1981. The various properties of the Cayley graphs are studied, see more in [1]. The concept of the Cayley graphs is also extended to many other abstract structures. In [2], Cayley graphs of groupoids, quasigroups, loops and groups are considered. Many properties of Cayley graphs of semigroups are examined in many papers, see for examples [3], [4], [5], [6], [7], [8], [9].

The characterizations, vertex-transitivities, color-preserving vertex-transitivities and undirected (symmetric) properties of Cayley graphs of some semigroups were must studied together. In order to investigate Cayley graphs of semigroups, it is first of all interesting to find the analogues of natural conditions which have been used in the group case. For example, it is well known that all Cayley graphs of groups are vertex-transitive and the Cayley graph $\mathrm{Cay}(G, A)$ of a group G is undirected if and only if $A = A^{-1}$.

In 2002, Kelarev [6] characterized the undirected Cayley graph of all periodic (and, therefore, all finite) semigroups. The next year, Kelarev and Praeger [8] investigated transitivity properties of Cayley graphs of semigroups in general. In the bottom, they posed a question: Is it true that if S is a semigroup with a subset A such that $\mathrm{Cay}(S, \{a\})$ is $\mathrm{Aut}(S, \{a\})$-vertex-transitive, for all $a \in A$, then $\mathrm{Cay}(S, A)$ is $\mathrm{ColAut}(S, A)$-vertex-transitive, too? One year later, Jiang [10] showed that in general the answer is negative and it is positive in the cases of bands and completely simple semigroups. Thereafter, there are many studies about properties of Cayley graphs of some special semigroups. Arworn et al. [3] characterized digraphs of right(left) zero unions of groups in 2003. Panma et al. [11] characterized Clifford semigroups digraphs in 2006. The next year, Fan and Zeng [5] considered all vertex-transitive Cayley graphs of bands together with undirected property. In 2009, Khosravi [12] characterized the vertex-transitive properties of Cayley graphs of left groups and also extended some results to direct product of a band and a group. Panma et al. [13] investigated Cayley graphs of strong semilattices of right and left groups to be vertex-transitive. The next year, the authors in [14] and [15] characterized rectangular groups digraphs. The undirected property was also studied in [14]. In the next year, Luo et al. [16] studied the basic structure and properties of Cayley graphs of completely simple semigroups and gave necessary and sufficient conditions for the graphs to be a disjoint union of complete graphs. They also characterized the strongly connected components of the graphs which are bipartite. In the same year, those graphs were characterized by Meksawang et al. [17]. In 2013, Wang and Li gave necessary and sufficient conditions of all vertex-transitivities of Cayley graphs of completely 0-simple semigroups and the conditions to be undirected which can be found in [18]. Next, Suksumran and Panma in [19] considered the connectedness of Cayley graphs of semigroups and gave necessary and sufficient conditions for a Cayley graph of a semigroup to be strongly connected and weakly connected in 2015. Later in 2016, Khosavi [9] gave descriptions for all vertex transitivities of Cayley graphs of cancellative semigroups.

A full transformation semigroup is a semigroup which contains all functions from a set X into itself with composition as the semigroup operation, denoted by $T(X)$. It is analogous to a permutation group, namely a transformation semigroup which contains identity and its elements are invertible is a permutation group. Furthermore, Cayley's theorem in group theory states that every group G is isomorphic to a subgroup of a symmetric group of G, a permutation group, and in semigroup theory, asserts that every semigroup is isomorphic to a subsemigroup of some full transformation semigroup. Therefore, the semigroups are specialize studying.

In 1966, Magill introduced and studied the subsemigroup $F(X,Y) = \{\alpha \in T(X) : Y\alpha \subseteq Y\}$ of $T(X)$, where $\emptyset \neq Y \subseteq X$. Later in 1975, Symons [20] introduced the full transformation semigroup with restricted range, $T(X,Y)$, which is a subsemigroup of $T(X)$ consisting of elements whose range is contained in Y. In that work, he described the automorphisms of $T(X,Y)$. We note that $T(X,X) = T(X)$ and hence $T(X,Y)$ is a generalization of $T(X)$. Regular elements of $T(X,Y)$ were characterized in [21] by Nenthein et al. in 2005. Recently, Sanwong and Sommanee [22] determined all Green's relations on $T(X,Y)$.

In this dissertation, we consider about Cayley graphs of semigroups of transformation with restricted range. Our main results are the characterizations of Cayley graphs of finite transformation semigroups with restricted range, study the connectedness: strongly connected, unilaterally connected and weakly connected, give conditions to be vertex-transitive and color-preserving vertex-transitive of the graphs. Moreover, the undirected property of the graphs are considered.