CHAPTER 2

Preliminaries

In this chapter, we address some necessary basic definitions and notations. We split this chapter into four sections, semigroups, transformation semigroups, digraphs and Cayley digraphs, and vertex-transitivities.

2.1 Semigroups

The basic definitions of semigroups are taken from [23], [24] and [25].

Definition 2.1.1. Let S be a non-empty set. A *binary operation* on S is a function $*$ from $S \times S$ to S with $(x, y) \mapsto x * y$ and a pair $(S, *)$ is called a *groupoid*. A *semigroup* is a groupoid $(S, *)$ where $*$ is associative on S, i.e., $x * (y * z) = (x * y) * z$ for all $x, y, z \in S$.

We abbreviate $(S, *)$ by S and write simply xy instead of $x * y$.

Definition 2.1.2. Let S be a semigroup. An element 1 of S is called an *identity* of S if $x1 = x = 1x$ for all $x \in S$ and that S is called a *semigroup with identity* or a *monoid*. A monoid G is a *group* if for each $a \in G$, there exists $b \in G$ such that $ab = 1 = ba$. An element z of S is called a *zero* if $zx = z = xz$ for all $x \in S$ and then S is called a *semigroup with zero.*

Definition 2.1.3. Let A be a non-empty subset of a semigroup S. We say that A is a *subsemigroup* of S if A is closed under the binary operation of S. A subsemigroup of S which is a group will be called a *subgroup* of S.

Definition 2.1.4. Let S and T be semigroups. A mapping $\phi : S \to T$ is called a *homomorphism* if

$$(xy)\phi = (x\phi)(y\phi)$$

for all $x, y \in S$.

We refer to S as the *domain* of ϕ and T as the *codomain* of ϕ. The subset

$$S\phi = \{x\phi \ : \ x \in S\}$$

of T is called the *range* (or *image*) of ϕ and sometimes denoted by $\mathrm{im}(\phi)$.

Definition 2.1.5. Let S and T be semigroups and ϕ a homomorphism from S into T. If ϕ is one-one, then it is called a *monomorphism* and S is *embedded* in T. If ϕ is onto, it is an *epimorphism*. ϕ is called an *isomorphism* if it is one-one and onto. A homomorphism from S into S is called an *endomorphism*, and if it is bijective it is called *automorphism*. If there exists an isomorphism ϕ from S onto T, we say that S and T are *isomorphic* and write $S \cong T$.

Definition 2.1.6. The *direct product* $S \times T$ of semigroups S and T is a semigroup with the binary operation defined by

$$(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2)$$

for all $s_1, s_2 \in S, t_1, t_2 \in T$.

Definition 2.1.7. An element x of a semigroup S is said to be *periodic* if there are positive integers m, n such that $x^{m+n} = x^m$. S is called a *periodic semigroup* if every element of S is periodic.

Definition 2.1.8. An element a of a semigroup S is said to be *regular* if there exists $x \in S$ such that $a = axa$ and S will be called a *regular semigroup* if all elements in S are regular.

Definition 2.1.9. For a non-empty subset A of a semigroup S, the *subsemigroup generated by* A, denoted by $\langle A \rangle$, is the subsemigroup of S containing of the elements that can be expressed as finite products of elements in A, i.e.,

$$\langle A \rangle = \{a_1 a_2 \cdots a_n \ : \ a_1, a_2, \ldots, a_n \in A \text{ and } n \in \mathbb{N}\}.$$

In particular, if A is a finite set, namely $A = \{a_1, a_2, \ldots, a_n\}$, we write

$$\langle A \rangle = \langle a_1, a_2, \ldots, a_n \rangle.$$

Especially, if $A = \{a\}$, a singleton set, then

$$\langle a \rangle = \{a, a^2, a^3, \ldots\}.$$

Definition 2.1.10. A subset A of a semigroup S is said to be a *set of generators* or a *generating set* of S if $\langle A \rangle = S$.

Definition 2.1.11. Let A and B be any non-empty subsets of a semigroup S. Put

$$AB = \{ab \ : \ a \in A, b \in B\}.$$

When dealing with singleton sets we shall use the notational simplifications that are usual in algebra, for example, writing Ab instead of $A\{b\}$. For any $x \in S$, xA and Ax is called a *left coset* and *right coset* of A, respectively.

Definition 2.1.12. A non-empty subset A of a semigroup S is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$ and an (two-sided) *ideal* if it is both a left and right ideal. An ideal of a semigroup S is *proper* if it is not equal to S.

Obviously, every ideal is a subsemigroup but some subsemigroups are not ideals.

Let $(S, *)$ be a semigroup which has no an identity element. Define a set $S \cup \{1\}$ under a binary operation $\cdot$ by

$$1 \cdot 1 = 1, 1 \cdot x = x = x \cdot 1 \text{ and } x \cdot y = x * y \text{ for all } x, y \in S.$$

We see that $S \cup \{1\}$ is a monoid with the identity element 1. Let S be a semigroup. The monoid S^1 is defined by

$$S^1 = \begin{cases} S & \text{if S has an identity element,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

We refer to S^1 as the *monoid obtained from S by adjoining an identity if necessary.*

If a is an element of a semigroup S without the identity, Sa need not contain a.

Remark 2.1.13.

$$S^1a = Sa \cup \{a\},$$
$$aS^1 = aS \cup \{a\},$$
$$S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}.$$

Definition 2.1.14. Let S be a semigroup and $a \in S$. The set S^1a (aS^1 and S^1aS^1) is called the *principal left ideal (right ideal and ideal, respectively) generated by a.*

Definition 2.1.15. Let S be a semigroup and $a, b \in S$. The equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ on S are defined by

1. $a\mathcal{L}b$ if and only if $S^1a = S^1b$;

2. $a\mathcal{R}b$ if and only if $aS^1 = bS^1$;

3. $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$;

4. $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

The relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ are called *Green's relations* on S.

It is immediate that $\mathcal{H} \subseteq \mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. In a group G,

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = G \times G.$$

Since the Green's relations are equivalence relations, we define the notations about equivalence classes for convenience: the $\mathcal{L}$-class containing a will be denoted by L_a. Likewise, the $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing a are denoted by R_a, H_a, D_a and J_a, respectively.

Proposition 2.1.16 ([23])**.** *If S is a periodic semigroup, then $\mathcal{D} = \mathcal{J}$.*

Definition 2.1.17. A semigroup S without zero is said to be *left simple* if it has no proper left ideals, S is *right simple* if it has no proper right ideals and S is *simple* if it has no proper ideals.

Theorem 2.1.18. *Let S be a semigroup. The following statements hold:*

1. *S is left simple if and only if $Sa = S$ for all $a \in S$;*
2. *S is right simple if and only if $aS = S$ for all $a \in S$;*
3. *S is simple if and only if $SaS = S$ for all $a \in S$.*

Definition 2.1.19. An element e in a semigroup S is an *idempotent* if $e^2 = e$. If all elements in S are idempotents, S is called a *band* or an *idempotent semigroup.*

Definition 2.1.20. A nonzero idempotent e in a semigroup S is called *primitive* if e is minimal in the set of all nonzero idempotents of S with respect to the following partial order on the set of all idempotents of S, i.e., $e \leq f$ if and only if $ef = fe = e$.

Definition 2.1.21. A semigroup without zero is *completely simple* if it contains only one element or it is simple and contains a primitive idempotent.

Definition 2.1.22. Let G be a group, I and Λ non-empty sets, and let $P = (p_{\lambda i})$ be a $(\Lambda \times I)$-matrix with entries $p_{\lambda i} \in G$ for all $\lambda \in \Lambda, i \in I$. The *Rees matrix semigroup* $M(G; I, \lambda; P)$ over G with *sandwich matrix* P consists of all triples (i, g, λ), where $i \in I, \lambda \in \Lambda$, and $g \in G$, with multiplication defined by

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu),$$

for all $a, b \in G, i, j \in I, \lambda, \mu \in \Lambda$.

Theorem 2.1.23 ([23]). (*The Rees Theorem*) *Every completely simple semigroup is isomorphic to a Rees matrix semigroup $M(G; I, \lambda; P)$ over a group G.*

Theorem 2.1.24 ([23]). *Let S be a semigroup without zero. Then the following conditions are equivalent:*

1. *S is completely simple;*

2. *S is regular, and has the weak cancellation property: for all $a, b, c \in S$,*

$$if\ ca = cb \text{ and } ac = bc,\ imply\ a = b;$$

3. *S is regular, and for all $a \in S$*

$$if\ aba = a,\ implies\ bab = b;$$

4. *S is regular and every idempotent is primitive.*

Theorem 2.1.25. *Every finite simple semigroup is a completely simple semigroup.*

Definition 2.1.26. A semigroup S is said to be the *left cancellative semigroup* if $ab = ac$ implies $b = c$ and the *right cancellative semigroup* if $ba = ca$ implies $b = c$ for all $a, b, c \in S$.

Definition 2.1.27. A semigroup S is called a *left zero semigroup* if $xy = x$ and a *right zero semigroup* if $xy = y$ for all $x, y \in S$.

Definition 2.1.28. A semigroup S is called a *left group* if S is left simple and right cancellative. Similarly, S is called a *right group* if S is right simple and left cancellative.

Theorem 2.1.29 ([24]). *Let S be a semigroup. Then the following statements are equivalent:*

1. *S is a left (right) group;*

2. *S is a left (right) simple semigroup which contains an idempotent;*

3. *S is isomorphic to the direct product of a left (right) zero semigroup and a group.*

Theorem 2.1.30 ([24]). *For any periodic semigroup S, the following statements are equivalent:*

1. *S is left (right) simple;*

2. *S is a left (right) group;*

3. *S is isomorphic to the direct product of a left (right) zero band and a group.*

2.2 Transformation Semigroups

Definition 2.2.1. Let X be any non-empty set. A *partial transformation semigroup on* X is the set of functions from a subset of X into X under composition which is denoted by $P(X)$. A *transformation* on X is a function from X into itself. The *full transformation semigroup* on X is the set of all transformations on X under composition and denoted by $T(X)$.

It is well known that $P(X)$ and $T(X)$ are regular semigroups with identity.

Definition 2.2.2. A transformation which is one-one and onto is called a *permutation*. The set of all permutations from X onto X is called the *symmetric group* and denoted by $G(X)$. A *permutation group* is a subgroup of the symmetric group.

By the simple combinatorial arguments, we obtain that if $|X| = n$, then $|T(X)| = n^n$ and $|G(X)| = n!$.

Definition 2.2.3. For $\alpha \in T(X)$, $x \in X$ and $Z \subseteq X$, the notation $x\alpha$ means that the *image* of x under α and let $Z\alpha$ denote the set of all images of elements in Z under α. The *rank* of α is the cardinal number of $\operatorname{im}(\alpha)$ and denoted by $\operatorname{rank}(\alpha)$. The *kernel* of α is given by

$$\ker(\alpha) = \{(a, b) \in X \times X \mid a\alpha = b\alpha\}.$$

The symbol π_α denotes the partition of X induced by α, namely

$$\pi_\alpha = \{y\alpha^{-1} : y \in \operatorname{im}(\alpha)\}$$

where $y\alpha^{-1}$ is the set of all $x \in X$ such that $x\alpha = y$.

It is easy to check that for any $\alpha, \beta \in T(X)$,

$$\ker(\alpha) = \ker(\beta) \text{ if and only if } \pi_\alpha = \pi_\beta. \tag{2.1}$$

Definition 2.2.4. For $a \in X$, the *constant map* σ_a in $T(X)$ is the transformation which $\operatorname{im}(\sigma_a) = \{a\}$.

Theorem 2.2.5 ([24])**.** *For $\phi \in T(X)$, ϕ is an idempotent if and only if $x\phi = x$ for all $x \in \operatorname{im}(\phi)$.*

Theorem 2.2.6 ([24])**.** *Let $\alpha, \beta \in T(X)$. Then*

1. *$\alpha\mathcal{L}\beta$ if and only if* $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$*;*

2. $\alpha\mathcal{R}\beta$ *if and only if* $\ker(\alpha) = \ker(\beta)$*;*

3. $\alpha\mathcal{D}\beta$ *if and only if* $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$*;*

4. $\mathcal{D} = \mathcal{J}$*.*

Theorem 2.2.7 ([24]). *Every semigroup can be embedded in some full transformation semigroup.*

By the above theorem, the transformation semigroups are very consequential in semigroup theory.

Definition 2.2.8. For a non-empty subset Y of X, the *full transformation semigroup with restricted range*

$$T(X,Y) = \{\alpha \in T(X) : \mathrm{im}(\alpha) \subseteq Y\}$$

is a subsemigroup of $T(X)$.

Obviously, $T(X,X) = T(X)$. Thus $T(X,Y)$ can be considered as a generalization of $T(X)$.

The set

$$F(X,Y) = \{\alpha \in T(X,Y) : X\alpha = Y\alpha\}$$

consisting of all regular elements in $T(X,Y)$ is the largest regular subsemigroup of $T(X,Y)$ ([22]).

Lemma 2.2.9 ([26]). *Let* $\alpha, \beta \in T(X,Y)$*. Then* $X\beta \subseteq Y\alpha$ *if and only if there exists* $\gamma \in T(X,Y)$ *such that* $\gamma\alpha = \beta$*.*

Theorem 2.2.10 ([22]). *Let* $\alpha, \beta \in T(X,Y)$*. Then* $\beta = \alpha\mu$ *for some* $\mu \in T(X,Y)$ *if and only if* $\ker(\alpha) \subseteq \ker(\beta)$*.*

The Green's relations on $T(X,Y)$ were studied by Sanwong and Sommanee [22] in 2008 as follows.

Theorem 2.2.11 ([22]). *Let* $\alpha, \beta \in T(X,Y)$*.*

1. $\alpha\mathcal{L}\beta$ *if and only if* ($\alpha, \beta \in F(X,Y)$ *and* $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$)
   or ($\alpha, \beta \in T(X,Y)\backslash F(X,Y)$ *and* $\alpha = \beta$)*;*

2. $\alpha\mathcal{R}\beta$ *if and only if* $\ker(\alpha) = \ker(\beta)$*;*

3. $\alpha\mathcal{D}\beta$ *if and only if* $(\alpha, \beta \in F(X,Y)$ *and* $\text{rank}(\alpha) = \text{rank}(\beta))$
or $(\alpha, \beta \in T(X,Y)\backslash F(X,Y)$ *and* $\ker(\alpha) = \ker(\beta))$*;*

4. $\alpha\mathcal{J}\beta$ *if and only if* $\ker(\alpha) = \ker(\beta)$ *or* $\text{rank}(\alpha) = |Y\alpha| = |Y\beta| = \text{rank}(\beta)$*.*

Lemma 2.2.12 ([20])**.** *Let X be a finite set and Y_1, Y_2 be non-empty subsets of X. Then $T(X, Y_1) \cong T(X, Y_2)$ if and only if $|Y_1| = |Y_2|$.*

By the above lemma, there is no loss of generality in assuming $X = X_n = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, r\}$. For convenience, we use the notations T_n and $T_{n,r}$ instead of $T(X)$ and $T(X,Y)$, respectively. Again by combinatorial arguments, we have that $|T_{n,r}| = r^n$.

The notation $\alpha = \begin{pmatrix} X_1 & X_2 & \ldots & X_k \\ y_1 & y_2 & \ldots & y_k \end{pmatrix}$ means to $X = \dot{\bigcup}_{i=1}^{k} X_i$, $y_i \in \text{im}(\alpha)$, $y_i\alpha^{-1} = X_i$ for all $i = 1, 2, \ldots, k$ and the least element in X_i less than the least element in X_{i+1} for all $i = 1, 2, \ldots, k-1$. For convenience, if $\alpha \in T(X)$ where $\alpha = \begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$, we write $\alpha = [a_1, a_2, \ldots, a_n]$.

Definition 2.2.13. An element ε in $T(X,Y)$ is called an *identity on* Z, a non-empty subset of Y, if $\text{im}(\varepsilon) = Z$ and $z\varepsilon = z$ for all $z \in Z$.

Definition 2.2.14. For $\alpha \in T(X,Y)$ and a non-empty subset Z of X, α is called a *permutation on* Z if $\alpha_{|_Z}$, the restriction of α to Z, is one-one and $Z\alpha = Z$.

By [27], we obtain the following lemma.

Lemma 2.2.15 ([27])**.** *Let $Y \subseteq X$ and $\alpha \in T(X,Y)$. If $Y\alpha = Y$, then $T(X,Y) = T(X,Y)\alpha$.*

Definition 2.2.16. Let C be a collection of sets. A *transversal* is a set which contains exactly one element from each member of the collection.

Proposition 2.2.17 ([28])**.** *Let $\emptyset \neq A \subseteq T(X)$. Then $\langle A \rangle$ is a completely simple semigroup if and only if for all $\alpha, \beta \in A$, $\text{im}(\alpha)$ is a transversal of π_β.*

Definition 2.2.18. The *number of k-combination* from a set of n elements, usually denoted by $\binom{n}{k}$, is equal to

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Given nonnegative integers n and k, the *Stirling number of the second kind*, denoted by $S(n,k)$, is the number of ways to partition a set of n objects into k non-empty subsets.

The recurrence relation is

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)$$

for $n, k > 0$ with initial conditions $S(0,0) = 1$ and $S(0,n) = 1 = S(n,0)$ and the explicit formula is

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \begin{pmatrix} k \\ i \end{pmatrix} (k-i)^n.$$

The Table 2.1 shows the values for the Stirling numbers of the second kind with $k \geq 0$.

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3025	7770	6951	2646	462	36	1	
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

Table 2.1: The values of $S(n,k)$ where $0 \leq k \leq n \leq 10$

From Theorem 2.2.11 and (2.1), we get for $\alpha, \beta \in T(X,Y)$, $\alpha\mathcal{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$. This means α and β are in the same $\mathcal{R}$-class if and only if $\pi_\alpha = \pi_\beta$. Therefore,

$$S(n,k) = \text{ the number of } \mathcal{R}\text{-classes of } T(X,Y) \text{ with rank } k.$$

2.3 Digraphs and Cayley Digraphs

In this part, we present about basic knowledge of digraphs and Cayley digraphs adduced from [1], [29] and [30].

Definition 2.3.1. A *directed graph* or *digraph* $D = (V,E)$ composes a *vertex* set V together with an *edge* set or *arc* set $E \subseteq V \times V$. For the edge $e = (u,v)$ of E, u is called the *tail* of e, and v is the *head* of e. If $u = v$, the edge (u,v) is said to be a *loop*.

In a drawing of a digraph, the direction of an edge is indicated with an arrow.

Definition 2.3.2. Let u and v be vertices of the digraph D. A (u,v)-*semi-diwalk* in D is a sequence

$$u = u_0, u_1, u_2, \ldots, u_k = v$$

of vertices, beginning with u and ending at v, such that (u_{i-1}, u_i) or (u_i, u_{i-1}) is an edge for $i = 1, 2, \ldots, k$. The number of edges in a semi-diwalk is called the *length* of the semi-diwalk. A (u, v)-semi-diwalk is called (u, v)-*diwalk* if (u_{i-1}, u_i) is an edge for all i. A (u, v)-*semi-dipath* is a (u, v)-semi-diwalk in which no vertex is repeated, and a (u, v)-*dipath* is a (u, v)-diwalk in which no vertex is repeated. A *dicycle* is a diwalk in which no vertex is repeated except for the beginning and ending vertices. An *n-dicycle* is a dicycle of length n and denoted by C_n.

Definition 2.3.3. Let D be a digraph and u, v be distinct vertices in D. The digraph D is *strongly connected* if a (u, v)-dipath exists. It is *unilaterally (one-sided) connected* if a (u, v)-dipath or (v, u)-dipath exists. It is called *weakly connected* (or *connected*) if a (u, v)-semi-dipath exists. A maximal connected subgraph of a digraph D is called a *component* of D.

Definition 2.3.4. A subgraph H of a digraph D is called an *induced subgraph* of D if whenever $u, v \in V(H)$ and $(u, v) \in E(D)$, then $(u, v) \in E(H)$. Let A be a non-empty set of vertices of a digraph D. The *subgraph of D induced by A* is the induced subgraph with vertex set A and denoted by $D[A]$ or simply $[A]$. For a positive integer n, the digraph nD is the union of n disjoint copies of D.

Definition 2.3.5. Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. A mapping $\phi : V_1 \to V_2$ is called a *(digraph) homomorphism* if $(u\phi, v\phi) \in E_2$ for all $(u, v) \in E_1$, i.e., ϕ preserves edges and denoted by $\phi : D_1 \to D_2$. Let D be a digraph. A digraph homomorphism $\phi : D \to D$ will be called an *(digraph) endomorphism.* If $\phi : D_1 \to D_2$ is a bijective homomorphism and ϕ^{-1} is also a homomorphism, then ϕ is called an *(digraph) isomorphism.* A digraph isomorphism $\phi : D \to D$ will be called an *(digraph) automorphism.*

We will denote by $\mathrm{End}(D)$ the set of all endomorphisms of D and $\mathrm{Aut}(D)$ the set of all automorphisms of D.

Definition 2.3.6. Let S be a semigroup and $A \subseteq S$. The *Cayley digraph* $\mathrm{Cay}(S, A)$ of a semigroup S with respect to A (which is simply called *Cayley graph*) is defined as the digraph with vertex set S and edge set $E(\mathrm{Cay}(S, A))$ consisting of those ordered pairs (x, y) such that $y = xa$ for some $a \in A$. The set A is called the *connection set* of $\mathrm{Cay}(S, A)$.

Clearly, if A is an empty set, $\mathrm{Cay}(S, A)$ is an empty graph. Therefore in the sequel, we suppose that the connection set A is a non-empty set.

Definition 2.3.7. A digraph $D = (V, E)$ is said to be *undirected* if, for each $(u, v) \in E$, the edge (v, u) belongs to E.

In a drawing of an undirected graph, the direction of edges are not given. The next proposition characterizes some properties of Cayley digraphs of groups.

Proposition 2.3.8 ([31]). *For a group G and a subset A of G, the following hold:*

1. $\mathrm{Cay}(G, A)$ *is weakly connected if and only if* $\langle A \rangle = G$*;*

2. $\mathrm{Cay}(G, A)$ *is undirected if and only if* $A = A^{-1}$ *where* $A^{-1} = \{x^{-1} : x \in A\}$.

The undirected Cayley digraphs of periodic semigroups were characterized by Kelarev [6] as follows.

Lemma 2.3.9 ([6]). *Let S be a semigroup with a subset A such that $\langle A \rangle$ is a periodic subsemigroup. The the following conditions are equivalent:*

1. *the Cayley digraph* $\mathrm{Cay}(S, A)$ *is undirected;*

2. *$SA = S$, the semigroup $\langle A \rangle = M(G; I, \Lambda; P)$ is completely simple and, for each $(i, a, \lambda) \in A$ and $j \in I$, there exists $\mu \in \Lambda$ such that $(j, p_{\lambda j}^{-1} a^{-1} p_{\mu i}^{-1}, \mu) \in A$.*

Theorem 2.3.10 ([6]). *For a periodic semigroup S, the following statements are equivalent:*

1. *there exists a subset A of S such that* $\mathrm{Cay}(S, A)$ *is undirected;*

2. *S has a completely simple subsemigroup C such that $SC = S$.*

Now, we introduce a new definition which is used in this dissertation.

Definition 2.3.11. The *representative graph* (graphical presentation, in [32]) G_α of a transformation $\alpha \in T(X, Y)$ is a digraph with the vertex set $V(G_\alpha) = X$ and the edge set $E(G_\alpha) = \{(x, y) \in X \times X : x\alpha = y\}$. For $A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ which is a subset of $T(X, Y)$, a representative graph G_A is a digraph with $V(G_A) = X$ and $E(G_A) = \{(x, y) \in E(G_\alpha) : \alpha \in A\}$.

Example 2.3.12. Let $n = 13$ and let α be the map

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 2 & 3 & 4 & 5 & 6 & 7 & 4 & 7 & 2 & 11 & 12 & 10 & 13 \end{pmatrix}. \tag{2.2}$$

Then the representative graph G_α can be drawn as in Figure 2.1.

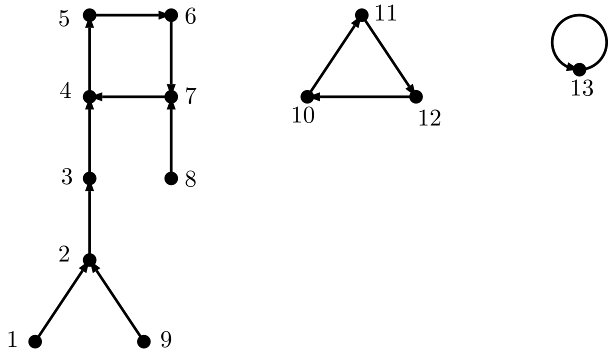

Figure 2.1: The representative graph G_α

In [32], Ayık et al. determined a new class for elements of $T(X)$, indeed it can be also defined for $T(X,Y)$. For $B = \{x_1, x_2, \ldots, x_m\} \subseteq X$, $\alpha \in T(X)$ is defined by

$$x_1\alpha = x_2, \ldots, x_{m-1}\alpha = x_m, x_m\alpha = x_r \text{ and } x\alpha = x \ (x \in X \backslash B)$$

where $x_r \in B$, α is called a *path-cycle* of *length* m and denoted by $\alpha = [x_1\ x_2 \ldots x_m \mid x_r]$. Moreover, if $m = r = 1$, α is called a *loop.* Hereafter, we let $s(\alpha)$ denote the set $B = \{x_1, x_2, \ldots, x_m\}$.

Furthermore, they described how to find a *linear notation* for elements of T_n by using the decomposition algorithm presented as follows.

Decomposition Algorithm.

Let $\alpha \in T_n$ be a non-identity map and $x \in X_n$. Then there is unique $n_x \in \mathbb{N}$ such that $x, x\alpha, \ldots, x\alpha^{n_x}$ are all distinct but $x\alpha^{n_x+1} \in \{x, x\alpha, \ldots, x\alpha^{n_x}\}$. Each path-cycle $\alpha_i = [x\ x\alpha\ \ldots\ x\alpha^{n_x} \mid x^{n_x+1}]$ is called a *divisor* of α.

1. α_1 is the *first factor* of α which is the divisor having the longest length. If there are divisors that the longest length is more than one, choose the one having the smallest first entry among them. Write each cycle $[x_1\ x_2\ \ldots\ x_k \mid x_1]$ where $x_1 = \min\{x_1, x_2, \ldots, x_k\}$.

2. Then define the *first residue* of α, $\alpha^{(1)}$, which is the function from X_n into itself given by

$$x\alpha^{(1)} = \begin{cases} x & \text{if } x \in s(\alpha_1), \\ x\alpha & \text{if } x \notin s(\alpha_1). \end{cases}$$

3. Carry out a similar procedure on $\alpha^{(1)}$, obtaining α_2 and $\alpha^{(2)}$, the *second factor* of α and the *second residue* of α, respectively. Continue this procedure until

$$\alpha_p \neq I, \alpha^{(p)} = I$$

where I is the identity function of T_n and $\alpha = \alpha_1\alpha_2\cdots\alpha_p$.

The integer p is called the *path-cycle rank of* α and denoted by $\mathrm{pcr}(\alpha)$. If α is the identity function, then $\mathrm{pcr}(\alpha) = 0$.

Example 2.3.13. From (2.2) in Example 2.3.12, the linear notation of α is the decomposition of the map into path-cycles as follows:

$$\alpha = [1\ 2\ 3\ 4\ 5\ 6\ 7 \mid 4]\ [10\ 11\ 12 \mid 10]\ [8\ 7 \mid 7]\ [9\ 2 \mid 2] \tag{2.3}$$

and then $\mathrm{pcr}(\alpha) = 4$.

In this dissertation, we will add all loops of the map to the right-side of the decomposition which does not affect to the decomposition. We call the number of path-cycles and loops of α that *path-cycle-loop rank* of α and denoted by $\mathrm{pclr}(\alpha)$.

Example 2.3.14. From the above example, we add $[13 \mid 13]$ into (2.3) at the right-side of the decomposition, hence that α is equal to

$$[1\ 2\ 3\ 4\ 5\ 6\ 7 \mid 4]\ [10\ 11\ 12 \mid 10]\ [8\ 7 \mid 7]\ [9\ 2 \mid 2]\ [13 \mid 13] \tag{2.4}$$

and $\mathrm{pclr}(\alpha) = 5$.

2.4 Vertex-Transitivities of Graphs

In this section, we give the definition of a vertex-transitive graph referred from [30]. Moreover, we address the characterizations of vertex-transitive Cayley digraphs of semigroups which are quoted from [8].

Definition 2.4.1. A digraph $D = (V, E)$ is said to be $\mathrm{Aut}(D)$-*vertex-transitive* or *vertex-transitive* ($\mathrm{End}(D)$-*vertex-transitive*) if for any two vertices $x, y \in V$, there exists an automorphism (endomorphism) ϕ such that $x\phi = y$. More generally, a subset C of $\mathrm{End}(D)$ is said to be *vertex-transitive on* D, and D is said to be C-*vertex-transitive* if, for any two vertices $x, y \in V$, there exists an endomorphism $\phi \in C$ such that $x\phi = y$.

Definition 2.4.2. For a Cayley digraph $\mathrm{Cay}(S, A)$, an element $\psi \in \mathrm{End}(\mathrm{Cay}(S, A))$ is said to be a *color-preserving endomorphism* if $xa = y$ implies $(x\psi)a = y\psi$, for all $x, y \in S$ and $a \in A$. Let us denote by $\mathrm{ColEnd}(S, A)$ and $\mathrm{ColAut}(S, A)$ the set of all color-preserving endomorphisms of $\mathrm{Cay}(S, A)$ and the set of all color-preserving automorphisms of $\mathrm{Cay}(S, A)$, respectively.

For a Cayley digraph $\mathrm{Cay}(S, A)$, we denote $\mathrm{End}(\mathrm{Cay}(S, A))$ by $\mathrm{End}(S, A)$ and $\mathrm{Aut}(\mathrm{Cay}(S, A))$ by $\mathrm{Aut}(S, A)$. Obviously,

$$\mathrm{ColAut}(S, A) \subseteq \mathrm{Aut}(S, A),$$

$$\mathrm{ColEnd}(S, A) \subseteq \mathrm{End}(S, A),$$

$$\mathrm{ColAut}(S, A) \subseteq \mathrm{ColEnd}(S, A),$$

and

$$\mathrm{Aut}(S, A) \subseteq \mathrm{End}(S, A).$$

It is well known that for every group G and $A \subseteq G$, $\mathrm{Cay}(G, A)$ is $\mathrm{Aut}(G, A)$-vertex-transitive. In 2003, Kelarev and Praeger [8] characterized all $\mathrm{Aut}(S, A)$-vertex-transitive and all $\mathrm{ColAut}(S, A)$-vertex-transitive Cayley digraphs of semigroups. They defined Cayley digraphs by the left action, i.e., (x, y) is an edge in the graph if $ax = y$ for some $a \in A$. For the right action, the theorems in this part are still true. Hence we give some results in the sense of right action.

Lemma 2.4.3 ([8]). *Let S be a semigroup with a subset A, let $s \in S$ and let C^s be the set of all vertices v of the Cayley digraph $\mathrm{Cay}(S, A)$ such that there is a dipath from s to v. Then C^s is equal to the left coset $s\langle A\rangle$.*

Lemma 2.4.4 ([8]). *Let S be a semigroup with a subset A such that $\langle A\rangle$ is completely simple and $SA = S$. Then every connected component of the Cayley digraph $\mathrm{Cay}(S, A)$ is strongly connected and, for each $v \in S$, the component containing v is $[v\langle A\rangle]$.*

Lemma 2.4.5 ([8]). *Let S be a semigroup and A a subset of S.*

1. *If the Cayley digraph $\mathrm{Cay}(S, A)$ is $\mathrm{End}(S, A)$-vertex-transitive, then $SA = S$.*

2. *If the Cayley digraph $\mathrm{Cay}(S, A)$ is $\mathrm{ColEnd}(S, A)$-vertex-transitive, then $Sa = S$ for each $a \in A$.*

The following two theorems are the characterizations of all $\mathrm{ColAut}(S, A)$-vertex-transitive and $\mathrm{Aut}(S, A)$-vertex-transitive Cayley digraphs such that all principal left ideals of the subsemigroup $\langle A\rangle$ are finite.

Theorem 2.4.6 ([8]). *Let S be a semigroup and let A be a subset of S such that all principal left ideals of the subsemigroup $\langle A\rangle$ are finite. Then the Cayley digraph $\mathrm{Cay}(S, A)$ is $\mathrm{ColAut}(S, A)$-vertex-transitive if and only if the following conditions hold:*

1. *$Sa = S$ for all $a \in A$;*

2. *$\langle A \rangle$ is isomorphic to a direct product of a left zero band and a group;*

3. *$|s\langle A \rangle|$ is independent of the choice of $s \in S$.*

Theorem 2.4.7 ([8])**.** *Let S be a semigroup and let A be a subset of S such that all principal left ideals of the subsemigroup $\langle A \rangle$ are finite. Then the Cayley digraph $\mathrm{Cay}(S, A)$ is $\mathrm{Aut}(S, A)$-vertex-transitive if and only if the following conditions hold:*

1. *$SA = S$;*

2. *$\langle A \rangle$ is a completely simple semigroup;*

3. *the Cayley digraph $\mathrm{Cay}(\langle A \rangle, A)$ is $\mathrm{Aut}(\langle A \rangle, A)$-vertex-transitive;*

4. *$|s\langle A \rangle|$ is independent of the choice of $s \in S$.*

Corollary 2.4.8 ([8])**.** *Let S be a finite rectangular band and A a subset of S. Then the Cayley digraph $\mathrm{Cay}(S, A)$ is $\mathrm{Aut}(S, A)$-vertex-transitive if and only if $A \cap sS \neq \emptyset$ for all $s \in S$.*

Given a family of digraphs $D_i = (V_i, E_i)$, where $i \in I$, their *union* is the digraph $D = \bigcup_{i \in I} D_i$ defined by

$$D = \Big(\bigcup_{i \in I} V_i, \bigcup_{i \in I} E_i\Big).$$

Lemma 2.4.9 ([8])**.** *Let S be a semigroup and A a subset of S. Then*

$$\mathrm{Cay}(S, A) = \bigcup_{a \in A} \mathrm{Cay}(S, \{a\}).$$

If $\mathrm{Cay}(S, A)$ is $\mathrm{ColAut}(S, A)$-vertex-transitive, then, for each $a \in A$, the Cayley digraph $\mathrm{Cay}(S, \{a\})$ is $\mathrm{ColAut}(S, \{a\})$-vertex-transitive.