CHAPTER 4

Conclusions

In Section 3.1, we addressed some properties of $T(X,Y)$. The important theorems are in the below.

1. $\langle A\rangle$ is a completely simple semigroup and $T(X,Y)A = T(X,Y)$ if and only if $Y\alpha = Y$ for all $\alpha \in A$.

2. If $Y\alpha = Y$ for all $\alpha \in A$, then $\langle A\rangle$ is a left simple semigroup.

In Section 3.2, we characterized the Cayley digraphs of $T(X,Y)$ as in the following. Let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq T(X,Y)$. Define a relation $\sim$ on $T(X,Y)$ by

$\beta \sim \gamma \;\Leftrightarrow\;$ there are representatives $\tau_1, \tau_2, \ldots, \tau_k$ of dicycles $C_{\tau_1}^{\alpha_{n_1}}, C_{\tau_2}^{\alpha_{n_2}}, \ldots, C_{\tau_k}^{\alpha_{n_k}}$ where $\alpha_{n_1}, \alpha_{n_2}, \ldots, \alpha_{n_k} \in A$, there exist $\beta_1, \beta_2, \ldots, \beta_k \in T(X,Y)$ such that $\beta = \beta_1, \gamma = \beta_k$ and $\beta_i\alpha_{n_i}^{s_i} = \tau_i = \beta_{i+1}\alpha_{n_i}^{t_i}$ for some $s_i, t_i \in \mathbb{N}, i = 1, 2, \ldots, k-1$.

Then the relation $\sim$ on $T(X,Y)$ is an equivalence relation and

$$\operatorname{Cay}(T(X,Y),A) \;\cong\; \dot{\bigcup_{\bar{\beta}\in T/\sim}} [\bar{\beta}].$$

In Section 3.3, we addressed the connectedness and undirected properties of the Cayley digraphs of $T(X,Y)$ which are shown in the following table.

Properties of Cay$(T(X,Y),A)$	**Necessary and Sufficient Conditions**
1. strongly connected	$\lvert Y\rvert = 1$.
2. weakly connected	$\sigma_z \in \langle A\rangle$ for some $z \in Y$.
3. unilaterally connected	3.1 $r = 1$, 3.2 $r = n = 2$ and $\{[2,1],[1,1]\} \subseteq A$, or 3.3 $r = n = 2$ and $\{[2,1],[2,2]\} \subseteq A$.
4. undirected	$Y\alpha = Y$ for all $\alpha \in A$ and $A_Y = (A_Y)^{-1}$.

In Section 3.4, we gave necessary and sufficient conditions being vertex-transitive and color-preserving vertex-transitive of Cayley digraphs of $T(X,Y)$. We obtained that the following statements are equivalent:

1. $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{Aut}(T(X,Y),A)$-vertex-transitive;

2. $\mathrm{Cay}(T(X,Y),A)$ is $\mathrm{ColAut}(T(X,Y),A)$-vertex-transitive;

3. $Y\alpha = Y$ for all $\alpha \in A$ and $|\beta\langle A\rangle|$ is independent of the choice of $\beta \in T(X,Y)$;

4. $\mathrm{Cay}(T(Y),A_Y)$ is $\mathrm{Aut}(T(Y),A_Y)$-vertex-transitive.