

CHAPTER 1

Introduction

Fractional calculus dates back to 1695 when Leibniz first suggested the possibility of fractional derivatives, which generalized the integer order of integration and differentiation to the arbitrary real or complex order. In recent years, fractional calculus has become an emerging field in mathematics and there are several different definitions of the fractional order derivative, such as the Riemann-Liouville definition [1], the Grünwald-Letnikov definition [1], and the Caputo definition [2]. The Caputo derivative is reformulated from the more classic Riemann-Liouville derivative and the initial conditions for Caputo's fractional differential equations are expressed in the same manner as for integer order differential equations [1]. In this study, we are interested in the Caputo derivative.

The fractional calculus has widely been applied in many fields, such as applied mathematics [3–5], physics [3,6], chemistry [5,7,8], biology [9–11], engineering [3,12,13], finance and economics [14]. Various studies formulated models using the fractional order system in the form of fractional differential equations [1,15–19] and fractional partial differential equations [20–23]. Fractional differential equations are power tools for describing many systems, for example, blood alcohol level, video tape counter readings and models for world population growth [24], obesity epidemics [25], Ebola epidemics [26], HIV/AIDS infection [27], and the spread of computer virus [28]. In addition, the results of fractional order systems may more closely resemble realistic dynamics than integer order systems [29].

There are so many scientific papers have applied fractional differential equations for population models, for instance, predator-prey models [30–33], interspecific competition models [34], bovine babesiosis disease and tick populations [35], SIS epidemic model [36], and mutualism models [37]. In this thesis, we focus on cooperative or mutualism models. Mutualism is used to represent an interaction between individuals of different species that benefit both of them. Facultative mutualism are ones whose populations are able to survive and reproduce without their mutualist [38,39], such as extrafloral nectaries plants and ants [40,41], and herbivorous crabs and coralline algae [42]. The two-species mutualism models in the form of ordinary differential equations are derived from modified Lotka-Volterra competition equations [39] and have been applied to a variety of ecological interactions [43–46].

In 1981, Robert May [47] presented the facultative mutualism model as follows:

$$\begin{cases} \frac{dx_1}{dt} = \hat{r}_1 x_1 \left[1 - \frac{x_1}{\hat{K}_1} + \hat{b}_{12} \frac{x_2}{\hat{K}_1} \right], \\ \frac{dx_2}{dt} = \hat{r}_2 x_2 \left[1 - \frac{x_2}{\hat{K}_2} + \hat{b}_{21} \frac{x_1}{\hat{K}_2} \right], \end{cases} \quad (1.0.1)$$

where x_i denotes the population density of i th species at time t , $\hat{r}_i$ is the intrinsic birth rate of i th species, $\hat{K}_i$ is the carrying capacity of the environment, $\hat{b}_{12}$ is the rate at which an individual of x_2 benefits the growth rate of population x_1 and in the same way, $\hat{b}_{21}$ represent the rate at which an individual of x_1 benefits the growth rate of population x_2 , that is, the mutualistic support the species give each other. Remark that all parameters are positive constants.

From the reference [39], the systems (1.0.1) in above is modified as

$$\begin{cases} \frac{dx_1}{dt} = \hat{r}_1 x_1 \left[1 - \frac{x_1}{\hat{K}_1 + \hat{b}_{12} x_2} \right], \\ \frac{dx_2}{dt} = \hat{r}_2 x_2 \left[1 - \frac{x_2}{\hat{K}_2 + \hat{b}_{21} x_1} \right]. \end{cases} \quad (1.0.2)$$

In 2012, Legovic and Gecek [48] modified the systems (1.0.1) and (1.0.2) by adding a harvesting effort specific to the population. Then the systems (1.0.1) and (1.0.2) become

$$\begin{cases} \frac{dx_1}{dt} = \hat{r}_1 x_1 \left[1 - \frac{x_1}{\hat{K}_1} + \hat{b}_{12} \frac{x_2}{\hat{K}_1} \right] - \hat{e}_1 x_1, \\ \frac{dx_2}{dt} = \hat{r}_2 x_2 \left[1 - \frac{x_2}{\hat{K}_2} + \hat{b}_{21} \frac{x_1}{\hat{K}_2} \right] - \hat{e}_2 x_2, \end{cases} \quad (1.0.3)$$

$$\begin{cases} \frac{dx_1}{dt} = \hat{r}_1 x_1 \left[1 - \frac{x_1}{\hat{K}_1 + \hat{b}_{12} x_2} \right] - \hat{e}_1 x_1, \\ \frac{dx_2}{dt} = \hat{r}_2 x_2 \left[1 - \frac{x_2}{\hat{K}_2 + \hat{b}_{21} x_1} \right] - \hat{e}_2 x_2, \end{cases} \quad (1.0.4)$$

where constants $\hat{e}_1$ and $\hat{e}_2$ are positive constants and represent the rate per capita of harvesting efforts on respective populations.

Motivated and inspired by the fractional calculus and the facultative mutualism models mentioned above, in this thesis we extend the systems (1.0.3) and (1.0.4) to fractional order forms.

Stability analysis is an important tool to understand system dynamics. There are many different methods that used to investigate stability of the system without visibly solving the equations [31, 49, 50]. Among them, the linearization and Lyapunov's direct methods are the popular technique for analysis of the local and global stability of system, respectively. The linearization method is applied to approximate the nonlinear system by using the linearized model [31, 51]. On the other hand, the Lyapunov's direct method is used to investigate the stability of a system based on Lyapunov function. In current years, many researchers proposed the Lyapunov's direct method to analyze the stability of fractional differential equations [49, 52–56]. Moreover, some Lyapunov functions have been constructed to study the stability of fractional differential equations [57–60]. However, the construction of the Lyapunov function and calculation of the fractional derivatives is complicated [54, 56].

In this thesis, we introduce two Caputo fractional order models for two-species facultative mutualism. Furthermore, the stability of the equilibrium points for the proposed models are analyzed to establish sufficient conditions for the local asymptotic stability of the non-coexistence equilibrium points. Likewise, we obtain sufficient conditions for the global uniform asymptotic stability of coexistence equilibrium points via the Lyapunov's direct method. Some numerical examples that illustrate the validity of the theoretical results are presented.