CHAPTER 2

Preliminaries

In this chapter, we give here the definitions, main theory, and some necessary notations that will be used throughout this thesis.

2.1 Special Functions of the Fractional Calculus

In this section, we introduce some basic theory of the special functions which are used in the fractional calculus. Some informations on the most important functions as the gamma function and the Mittag-Leffler function are provided. These functions play a crucial role in the theory of fractional differential equations.

Definition 2.1.1. (Gamma Function [1]**)** Let $z \in \mathbb{C}$. The gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad Re(z) > 0. \tag{2.1.1}$$

This function is generalization of a factorial in the following form:

$$\Gamma(z+1) = z\Gamma(z), \tag{2.1.2}$$

which can be easily proved by integrating by parts

$$\Gamma(z+1) = \int_0^{\infty} e^{-t} t^{z} dt = \Big[- e^{-t} t^{z} \Big]_{t=0}^{t=\infty} + z \int_0^{\infty} e^{-t} t^{z-1} dt = z\Gamma(z).$$

Obviously, $\Gamma(1) = 1$, and using (2.1.2) we obtained

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) = 1 = 1! \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1! = 2! \\
\vdots \quad & \quad \vdots \quad \quad \vdots \\
\Gamma(n+1) &= n \cdot \Gamma(n) = n \cdot (n-1)! = n!, \quad \text{where } n \in \mathbb{N}.
\end{aligned}$$

Definition 2.1.2. (Mittag-Leffler Function [1]**)** The Mittag-Leffler function is generalization of the exponential function. The one-parameter Mittag-Leffler function is defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \ z \in \mathbb{C}. \tag{2.1.3}$$

The two-parameter of the Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \ z \in \mathbb{C}. \tag{2.1.4}$$

2.2 Definition of the Caputo Fractional Derivative

In this section, we present the derivative of the Caputo definitions and give some necessary properties that are used in our study.

Definition 2.2.1. (Caputo Fractional Derivative [1]**)** For a function $f \in C^n$ given on the interval $[t_0, \infty]$. Suppose that $\alpha > 0$ and $t > t_0$ where $\alpha, t_0, t \in \mathbb{R}$, and $n \in \mathbb{N}$. Then

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{t_0}^{t} \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & , \ n-1 < \alpha < n, \\ \\ \dfrac{d^n}{dt^n} f(t) & , \ \alpha = n, \end{cases} \tag{2.2.1}$$

is called the Caputo fractional derivative of order α.

Remark 2.2.2. *([1]) For $\alpha \to n$ the Caputo derivative becomes a conventional n-th derivative of the function $f(t)$.*

Some properties of the Caputo differential operator which are most frequently used in applications are presented as the following.

(i) Linearity property ([19]) Let $f(t),\ g(t) : [a, b] \to \mathbb{R}$ be such that ${}_{t_0}^{C}D_t^{\alpha} f(t)$ and ${}_{t_0}^{C}D_t^{\alpha} g(t)$ exist and let $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, ${}_{t_0}^{C}D_t^{\alpha}(\lambda_1 f(t) + \lambda_2 g(t))$ exists, and

$$ {}_{t_0}^{C}D_t^{\alpha}(\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 \ {}_{t_0}^{C}D_t^{\alpha} f(t) + \lambda_2 \ {}_{t_0}^{C}D_t^{\alpha} g(t).$$

(ii) Caputo derivative of a constant ([1]) The Caputo derivative for a constant function $f(t) = c$ is zero, that is,

$$ {}_{t_0}^{C}D_t^{\alpha} c = 0.$$

2.3 Stability Theorems of Fractional Differential Equations

In the beginning of this section, we discuss some considerable definitions and theories that we use for studying stability of fractional order systems.

Definition 2.3.1. (Piecewise Continuous [61]**)** Let $f : [a, b] \to \mathbb{R}$ be such that it is continuous in $[a, b]$ except for a finite number of points, at each of which f has jump discontinuity. Then f is said to be piecewise continuous in $[a, b]$.

Definition 2.3.2. (Locally Lipschitz Continuous [62]**)** The function $f(t,x):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is said to be locally Lipschitz continuous in x if for some $h>0$ there exists $L>0$ such that

$$\|f(t,x_1)-f(t,x_2)\|\le L\|x_1-x_2\| \tag{2.3.1}$$

for all $x_1,\ x_2\in B_h=\{x\in\mathbb{R}^n\mid \|x\|<h\},\ t\ge t_0$. The constant L is called the Lipschitz constant. A definition for globally Lipschitz continuous functions follows by requiring the equation (2.3.1) to hold for $x_1,\ x_2\in\mathbb{R}^n$.

Theorem 2.3.3. *([63]) Let $\Omega\subset\mathbb{R}^n$. If a function $f(t,x):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be continuously differentiable on $[t_0,T]\times\Omega$ and assume that the derivative of f satisfies*

$$\left\|\frac{\partial f}{\partial x}(t,x)\right\|\le L \tag{2.3.2}$$

on $[t_0,T]\times\Omega$. The f is locally Lipschitz continuous on Ω with constant L.

Theorem 2.3.4. ***(Locally Bounded** [64]**)** If M is a metric space and $f:X\to M$ is a continuous function, then f is locally bounded.*

Definition 2.3.5. ([65]) A scalar continuous function $W(x)$ is said to be locally positive definite if $W(0)=0$ and in a ball $B_r=\{x\in\mathbb{R}^n\ |\|x\|<r\}$

$$x\ne 0\Rightarrow W(x)>0.$$

If $W(0)=0$ and the above property holds for the whole state space, then $W(x)$ is said to be globally positive definite.

Next, we move to the part of stability theory, which is used for analysis in our main study. Let's consider the following fractional order system

$${}_{t_0}^{C}D_t^{\alpha}x(t)=f(t,x), \tag{2.3.3}$$

with initial condition $x(t_0)$, where $\alpha\in(0,1)$, $f:[t_0,\infty)\times\Omega\to\mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0,\infty)\times\Omega$, and $\Omega\subset\mathbb{R}^n$ is a domain that contains the origin $x=0$.

Definition 2.3.6. ([49]) The constant x^* is an equilibrium point of Caputo fractional order system (2.3.3), if and only if $f(t,x^*)=0$.

Remark 2.3.7. *([49]) Any equilibrium point x^* can be shifted to the origin of $\mathbb{R}^n$; i.e. $x^*=0$ via a change of variables. Suppose the equilibrium point for (2.3.3) is $x^*\ne 0$ and consider the change of variable $y=x-x^*$. The αth order derivative of y is given by*

$${}_{t_0}^{C}D_t^{\alpha}y={}_{t_0}^{C}D_t^{\alpha}(x-x^*)={}_{t_0}^{C}D_t^{\alpha}x-{}_{t_0}^{C}D_t^{\alpha}x^*=f(t,x)=f(t,y+x^*)=g(t,y).$$

Since $f(t,x^*) = 0$ and $f(t,x) = g(t,y)$, where $y = x - x^*$. Replacing x by x^* then we have $g(t,0) = 0$ and in new variable y, the system has equilibrium at the origin.

Remark 2.3.8. *([1]) For the system (2.3.3), if $f(t,x)$ is the locally bounded and is locally Lipschitz with respect to x, then implies the existence and uniqueness of the solution to the Caputo fractional order system (2.3.3) on $[t_0, \infty) \times \Omega$.*

Definition 2.3.9. ([66]) For the system described by (2.3.3).
(i) $x^* = 0$ is said to be stable if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.
(ii) $x^* = 0$ is said to be asymptotically stable if it is stable and for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \varepsilon) > 0$ such that $\|x(t_0)\| < \delta_a$ implies $\lim\limits_{t\to\infty} \|x(t)\| = 0$.
(iii) $x^* = 0$ is said to be uniformly stable if it is stable and $\delta = \delta(\varepsilon) > 0$ can be chosen independently of t_0.
(iv) $x^* = 0$ is uniformly asymptotically stable if it is uniformly stable and there exists a $\delta_a > 0$, independent of t_0, such that, if $\|x(t_0)\| < \delta_a$ then $\lim\limits_{t\to\infty} \|x(t)\| = 0$.
(v) $x^* = 0$ is globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and δ_a can be an arbitrary large, finite number.

Theorem 2.3.10. *([37]) The equilibrium points x^* of system (2.3.3) are locally asymptotically stable if all eigenvalues λ_i, $i = 1, ..., n$ of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy: $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$.*

Definition 2.3.11. **(Lyapunov Function** [67]**)** Let $x^* = 0 \in \mathcal{D} \subset \mathbb{R}^n$ be an equilibrium point. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \qquad \forall x \in \mathcal{D}\backslash\{0\},$$
$$\dot{V} < 0, \qquad \forall x \in \mathcal{D}.$$

Then V is called a Lyapunov function.

Theorem 2.3.12. ***(Uniform Asymptotic Stability Theorem** [54]**)** Let $x^* = 0$ be an equilibrium point of the system (2.3.3) and $\Omega \subset \mathbb{R}^n$ be a domain containing $x^* = 0$. Let $V(t,x) : [t_0, \infty) \times \Omega \to \mathbb{R}$ be a continuously differentiable function such that*

$$W_1(x) \leq V(t,x) \leq W_2(x), \tag{2.3.4}$$

$$\text{and} \quad {}_{t_0}^{C}D_t^{\alpha} V(t,x) \leq -W_3(x), \tag{2.3.5}$$

with $\forall t \geq 0$, $\forall x \in \Omega$, and $0 < \alpha < 1$, where $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuous positive definite functions on Ω. Then $x^* = 0$ is uniformly asymptotically stable.

The above theorem applies to the local analysis of stability. In order to assert the global uniform asymptotic stability of a system, one might naturally expect that the ball B_r in the above local theorem has to be expanded to be the whole state space. This is indeed necessary, but it is not enough. An additional condition on the function V has to be satisfied: $V(x)$ must be radially unbounded, by which we mean that $V(x) \to \infty$ as $\|x\| \to \infty$ (in other words, as x tends to infinity in any directions) [65, 68].

We present some informations which are essentially used in analysis of the stability via the Lyapunov's direct method as the following.

Theorem 2.3.13. *([57]) Let $x(t) \in \mathbb{R}_+$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$*

$$ {}_{t_0}^{C}D_t^{\alpha} \left(x(t) - x^* - x^* \ln \frac{x(t)}{x^*} \right) \leq \left(1 - \frac{x^*}{x(t)} \right) {}_{t_0}^{C}D_t^{\alpha} x(t), \tag{2.3.6} $$

where $x^* \in \mathbb{R}_+$ and $\forall \alpha \in (0,1)$.

Lemma 2.3.14. *([69]) Suppose that $f(t) \in C[t_0, T]$, the solution to the Caputo differential equation*

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) = \lambda x(t) + f(t), \\ x(t_0) = x_{t_0}, \end{cases} \tag{2.3.7} $$

with $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$ has the form

$$ x(t) = x(t_0) E_{\alpha}(\lambda (t - t_0)^{\alpha}) + \int_{t_0}^{t} (t - t_0)^{\alpha - 1} E_{\alpha,\alpha}(\lambda (t - \tau)^{\alpha}) f(\tau) d\tau. $$

Lemma 2.3.15. *([70]) Let $x(t)$ be a continuous function on $[t_0, +\infty)$ and satisfying*

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) \leq \lambda x(t), \\ x(t_0) = x_{t_0}, \end{cases} \tag{2.3.8} $$

where $0 < \alpha < 1$, $\lambda \in \mathbb{R}$ and t_0 is the initial time. Then

$$ x(t) \leq x_{t_0} E_{\alpha}(\lambda (t - t_0)^{\alpha}). $$

Lemma 2.3.16. *Let $x(t)$ be a continuous function on $[t_0, +\infty)$ and satisfying*

$$ \begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) \geq \lambda x(t), \\ x(t_0) = x_{t_0}, \end{cases} \tag{2.3.9} $$

where $0 < \alpha < 1$, $\lambda \in \mathbb{R}$ and t_0 is the initial time. Then

$$ x(t) \geq x_{t_0} E_{\alpha}(\lambda (t - t_0)^{\alpha}). $$

Proof. There exists a nonnegative continuous function $m(t)$ satisfying

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x(t) = \lambda x(t) + m(t), \\ x(t_0) = x_{t_0}. \end{cases} \tag{2.3.10}$$

According to Lemma 2.3.14, the solution of the system (2.3.10) can be written as

$$x(t) = x_{t_0}E_{\alpha}(\lambda(t-t_0)^{\alpha}) + \int_{t_0}^{t}(t-t_0)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})m(\tau)d\tau, \quad t \geq t_0. \tag{2.3.11}$$

Since $E_{\alpha,\alpha}(x) > 0$ for $0 < \alpha < 1$ and $x \in \mathbb{R}$ [71], $m(t)$ is a nonnegative continuous function, it follows from (2.3.11) that

$$x(t) \geq x_{t_0}E_{\alpha}(\lambda(t-t_0)^{\alpha}), \quad t \geq t_0.$$

This complete the proof. □