CHAPTER 3

Model Description

In this chapter, we present alternative representations of the two-species facultative mutualism models with harvesting in the form of the Caputo fractional derivatives.

The system of ordinary differential equations for the two-species facultative mutualism model introduced by Robert May [47]. In this thesis, we are interested in two special cases of two-species facultative mutualism models, which are presented as the following.

A first model of facultative mutualism [72–75] with adding a harvesting effort specific to the population is described by a system of ordinary differential equations (1.0.3), which can be written in this form

$$\begin{cases} \dfrac{dx_1}{dt} = \widehat{r}_1 x_1 \left(1 - \dfrac{\widehat{e}_1}{\widehat{r}_1}\right) - \dfrac{\widehat{r}_1 x_1^2}{\widehat{K}_1} + \dfrac{\widehat{r}_1 \widehat{b}_{12} x_1 x_2}{\widehat{K}_1}, \\ \\ \dfrac{dx_2}{dt} = \widehat{r}_2 x_2 \left(1 - \dfrac{\widehat{e}_2}{\widehat{r}_2}\right) - \dfrac{\widehat{r}_2 x_2^2}{\widehat{K}_2} + \dfrac{\widehat{r}_2 \widehat{b}_{21} x_1 x_2}{\widehat{K}_2}. \end{cases} \tag{3.0.1}$$

The following system is another basic model [72, 74, 76] subject to proportional harvesting [48]

$$\begin{cases} \dfrac{dx_1}{dt} = \widehat{r}_1 x_1 \left(1 - \dfrac{\widehat{e}_1}{\widehat{r}_1}\right) - \dfrac{\widehat{r}_1 x_1^2}{\widehat{K}_1 + \widehat{b}_{12} x_2}, \\ \\ \dfrac{dx_2}{dt} = \widehat{r}_2 x_2 \left(1 - \dfrac{\widehat{e}_2}{\widehat{r}_2}\right) - \dfrac{\widehat{r}_2 x_2^2}{\widehat{K}_2 + \widehat{b}_{21} x_1}. \end{cases} \tag{3.0.2}$$

By the systems (3.0.1) and (3.0.2), we replace the integer order derivatives with fractional order Caputo derivatives. Then, we obtain the following generalized Caputo fractional models

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1 = \widehat{r}_1 x_1 \left(1 - \dfrac{\widehat{e}_1}{\widehat{r}_1}\right) - \dfrac{\widehat{r}_1 x_1^2}{\widehat{K}_1} + \dfrac{\widehat{r}_1 \widehat{b}_{12} x_1 x_2}{\widehat{K}_1}, \\ \\ {}_{t_0}^{C}D_t^{\alpha} x_2 = \widehat{r}_2 x_2 \left(1 - \dfrac{\widehat{e}_2}{\widehat{r}_2}\right) - \dfrac{\widehat{r}_2 x_2^2}{\widehat{K}_2} + \dfrac{\widehat{r}_2 \widehat{b}_{21} x_1 x_2}{\widehat{K}_2}, \end{cases} \tag{3.0.3}$$

and

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1 = \widehat{r}_1 x_1 \left(1 - \dfrac{\widehat{e}_1}{\widehat{r}_1}\right) - \dfrac{\widehat{r}_1 x_1^2}{\widehat{K}_1 + \widehat{b}_{12} x_2}, \\ \\ {}_{t_0}^{C}D_t^{\alpha} x_2 = \widehat{r}_2 x_2 \left(1 - \dfrac{\widehat{e}_2}{\widehat{r}_2}\right) - \dfrac{\widehat{r}_2 x_2^2}{\widehat{K}_2 + \widehat{b}_{21} x_1}. \end{cases} \tag{3.0.4}$$

Both of the systems (3.0.3) and (3.0.4) have some flaws as regards to the time dimension because the left-hand side has the dimension $(time)^{-\alpha}$, while the right-hand side has the dimension $(time)^{-1}$. The correct form of the systems (3.0.3) and (3.0.4) can be obtained as follows [77]:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1 = \widehat{r}_1^{\alpha} x_1 \left(1 - \dfrac{\widehat{e}_1^{\alpha}}{\widehat{r}_1^{\alpha}}\right) - \dfrac{\widehat{r}_1^{\alpha} x_1^2}{\widehat{K}_1} + \dfrac{\widehat{r}_1^{\alpha} \widehat{b}_{12} x_1 x_2}{\widehat{K}_1}, \\ \\ {}_{t_0}^{C}D_t^{\alpha} x_2 = \widehat{r}_2^{\alpha} x_2 \left(1 - \dfrac{\widehat{e}_2^{\alpha}}{\widehat{r}_2^{\alpha}}\right) - \dfrac{\widehat{r}_2^{\alpha} x_2^2}{\widehat{K}_2} + \dfrac{\widehat{r}_2^{\alpha} \widehat{b}_{21} x_1 x_2}{\widehat{K}_2}, \end{cases} \tag{3.0.5}$$

and

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1 = \widehat{r}_1^{\alpha} x_1 \left(1 - \dfrac{\widehat{e}_1^{\alpha}}{\widehat{r}_1^{\alpha}}\right) - \dfrac{\widehat{r}_1^{\alpha} x_1^2}{\widehat{K}_1 + \widehat{b}_{12} x_2}, \\ \\ {}_{t_0}^{C}D_t^{\alpha} x_2 = \widehat{r}_2^{\alpha} x_2 \left(1 - \dfrac{\widehat{e}_2^{\alpha}}{\widehat{r}_2^{\alpha}}\right) - \dfrac{\widehat{r}_2^{\alpha} x_2^2}{\widehat{K}_2 + \widehat{b}_{21} x_1}. \end{cases} \tag{3.0.6}$$

For convenience, we define the parameters, $r_i = \widehat{r}_i^{\alpha}$, $e_i = \widehat{e}_i^{\alpha}$, $K_i = \widehat{K}_i$, $b_{12} = \widehat{b}_{12}$ and $b_{21} = \widehat{b}_{21}$ where $i = 1, 2$. Then, we obtain the following modified systems

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1 = f_1(x_1, x_2) = r_1 x_1 \left(1 - \dfrac{e_1}{r_1}\right) - \dfrac{r_1 x_1^2}{K_1} + \dfrac{r_1 b_{12} x_1 x_2}{K_1}, \\ \\ {}_{t_0}^{C}D_t^{\alpha} x_2 = f_2(x_1, x_2) = r_2 x_2 \left(1 - \dfrac{e_2}{r_2}\right) - \dfrac{r_2 x_2^2}{K_2} + \dfrac{r_2 b_{21} x_1 x_2}{K_2}, \end{cases} \tag{3.0.7}$$

and

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1 = g_1(x_1, x_2) = r_1 x_1 \left(1 - \dfrac{e_1}{r_1}\right) - \dfrac{r_1 x_1^2}{K_1 + b_{12} x_2}, \\ \\ {}_{t_0}^{C}D_t^{\alpha} x_2 = g_2(x_1, x_2) = r_2 x_2 \left(1 - \dfrac{e_2}{r_2}\right) - \dfrac{r_2 x_2^2}{K_2 + b_{21} x_1}, \end{cases} \tag{3.0.8}$$

with initial conditions $x_1(t_0) = x_{10}$ and $x_2(t_0) = x_{20}$, where all the model parameters are assumed to be positive.

3.1 Existence and Uniqueness

The possible region of the models (3.0.7) and (3.0.8) is defined as the non-negative quadrant

$$\mathbb{R}^2_+ = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\right\}.$$

From the systems (3.0.7) and (3.0.8), it is clear that $f_i, g_i, \frac{\partial f_i}{\partial x_i}$ and $\frac{\partial g_i}{\partial x_i}$ for $i = 1, 2$ are continuous in $\mathbb{R}^2_+$. By following the Theorems 2.3.3 and 2.3.4, it can be deduced that $f = (f_1, f_2)$ and $g = (g_1, g_2)$ satisfies the local Lipschitz condition with respect to $x = (x_1(t), x_2(t))$ in $\mathbb{R}^2_+$ and locally bounded, respectively. Therefore, by Remark 2.3.8, the systems (3.0.7) and (3.0.8) have a unique solution in $\mathbb{R}^2_+$.

3.2 Non-Negative Solution

Theorem 3.2.1. *If $x_1(t_0) \geq 0$ and $x_2(t_0) \geq 0$, then there is a unique solution $x(t)$ to the Caputo fractional order model (3.0.7) on $t \geq t_0$ and the solution remains in $\mathbb{R}^2_+$.*

Proof. In section 3.1, a uniqueness solution of $x(t)$ to the system (3.0.7) is obtained. Thus, it only needs to be proved that the solution $x(t) = (x_1(t), x_2(t))$ remains in $\mathbb{R}^2_+$.

Let $x(t_0) = (x_1(t_0), x_2(t_0))$ in $\mathbb{R}^2_+$ be the initial solution of the system (3.0.7). By contradiction, suppose that there exists a solution $x(t)$ that lies outside of $\mathbb{R}^2_+$. The consequence is that $x(t)$ crosses the x_1 axis or x_2 axis. Now we have to consider two cases.

Case 1: If the solution $x(t)$ passes through the x_2 axis, then there exists t^* such that $t^* \geq t_0$ and $x_1(t^*) = 0$, and there exists t_1 sufficiently close to t^* such that $t_1 > t^*$ and $x_1(t) < 0$ for all $t \in (t^*, t_1]$. From the system (3.0.7), we have

$${}^{C}_{t_0}D^{\alpha}_{t} x_1 = r_1 x_1 \left(1 - \frac{e_1}{r_1}\right) - \frac{r_1 x_1^2}{K_1} + \frac{r_1 b_{12} x_1 x_2}{K_1}, \quad \text{for all } t \in [t^*, t_1].$$

The above expression may be written as

$${}^{C}_{t_0}D^{\alpha}_{t} x_1 = (r_1 - e_1 + \frac{r_1 b_{12} x_2}{K_1}) x_1 - N x_1^2, \quad \text{where } N = \frac{r_1}{K_1}. \tag{3.2.1}$$

Consider

$$\begin{aligned} {}^{C}_{t_0}D^{\alpha}_{t} x_2 &= r_2 x_2 \left(1 - \frac{e_2}{r_2}\right) - \frac{r_2 x_2^2}{K_2} + \frac{r_2 b_{21} x_1 x_2}{K_2} \\ &\leq r_2 x_2. \end{aligned} \tag{3.2.2}$$

Using Lemma 2.3.15, the solution $x_2(t)$ is

$$x_2(t) \leq x_2(t_0) E_\alpha(r_2 (t - t_0)^\alpha), \quad t \in [t^*, t_1]. \tag{3.2.3}$$

Define $M = x_2(t_0)E_\alpha(r_2(t-t_0)^\alpha)$, that is, $M \geq 0$ [71]. From the equation (3.2.1), we obtain

$$ {}_{t_0}^{C}D_t^\alpha x_1 \geq (r_1 - e_1 + \frac{r_1 b_{12}}{K_1}M)x_1 - N x_1^2. \tag{3.2.4}$$

Since t_1 can be chosen to be arbitrarily close to t^*, then $x_1(t) \leq -x_1^2(t)$ for all $t \in [t^*, t_1]$. So, the following is obtained:

$$ {}_{t_0}^{C}D_t^\alpha x_1 \geq (r_1 - e_1 + \frac{r_1 b_{12}}{K_1}M + N)x_1. \tag{3.2.5}$$

Using the result in Lemma 2.3.16 in the above inequality (3.2.5), then the solution is

$$x_1(t) \geq x_1(t_0)E_\alpha((r_1 - e_1 + \frac{r_1 b_{12}}{K_1}M + N)(t-t_0)^\alpha), \quad t \in [t^*, t_1]. \tag{3.2.6}$$

Thus, $x_1(t) \geq 0$ for any $t \geq t_0$, which contradicts the assumption.

Case 2: It is considered that the solution $x(t)$ passed through the x_1 axis. Because the second equation of the system (3.0.7) has the same form as the first equation; the proof for Case 2 is similar to the proof in the previous case.

Therefore, it can be concluded that the solution $x(t)$ of the system (3.0.7) lies within $\mathbb{R}_+^2$. □

Theorem 3.2.2. *If $x_1(t_0) \geq 0$ and $x_2(t_0) \geq 0$, then there is a unique solution $x(t)$ to the Caputo fractional order model* (3.0.8) *on $t \geq t_0$ and the solution remains in $\mathbb{R}_+^2$.*

Proof. From the section 3.1, we already get a uniqueness solution of $x(t)$ to the system (3.0.8). So, it suffices to prove that the solution $x(t) = (x_1(t), x_2(t))$ remains in $\mathbb{R}_+^2$.

Let $x(t_0) = (x_1(t_0), x_2(t_0))$ in $\mathbb{R}_+^2$ be the initial solution of the system (3.0.8). Proofing by contradiction, suppose that there exists a solution $x(t)$ that lies outside of $\mathbb{R}_+^2$. So, there are two possibilities, that is, $x(t)$ crosses the x_1 axis or x_2 axis. We shall then consider two cases.

Case 1: If the solution $x(t)$ passes through the x_2 axis, then there exists t^* such that $t^* \geq t_0$ and $x_1(t^*) = 0$, and there exists t_1 sufficiently close to t^* such that $t_1 > t^*$ and $x_1(t) < 0$ for all $t \in (t^*, t_1]$. From the system (3.0.8), we have

$$ {}_{t_0}^{C}D_t^\alpha x_1 = r_1 x_1 \left(1 - \frac{e_1}{r_1}\right) - \frac{r_1 x_1^2}{K_1 + b_{12}x_2}, \quad \text{for all } t \in [t^*, t_1]$$

and it follows that

$$ {}_{t_0}^{C}D_t^\alpha x_1 \geq (r_1 - e_1)x_1 - N x_1^2, \quad \text{where } N = \frac{r_1}{K_1}. \tag{3.2.7}$$

Since t_1 can be chosen to be arbitrarily close to t^*, then $x_1(t) \leq -x_1^2(t)$ for all $t \in [t^*, t_1]$. The following is obtained:

$$ {}_{t_0}^{C}D_t^{\alpha} x_1 \geq (r_1 - e_1 + N)x_1. \tag{3.2.8} $$

Using the Lemma 2.3.16, the solution is

$$ x_1(t) \geq x_1(t_0)E_{\alpha}((r_1 - e_1 + N)(t - t_0)^{\alpha}), \quad t \in [t^*, t_1]. \tag{3.2.9} $$

Thus, $x_1(t) \geq 0$ for any $t \geq t_0$, which contradicts the assumption.

Case 2: It is considered that the solution $x(t)$ passed through the x_1 axis. It is easy to see that the second equation of the system (3.0.8) has the same form as the first equation then the proof for Case 2 is similar to the proof in the previous case.

Hence, the solution $x(t)$ of the system (3.0.8) lies within $\mathbb{R}_+^2$. □