CHAPTER 4

Main Results

In this chapter, we investigate the stability of the equilibrium points of the models (3.0.7) and (3.0.8), which we called them the Model 1 and Model 2, respectively. The local and global stability of all equilibrium points are analyzed using the linearization method and the Lyapunov's direct method.

4.1 Model 1

The equilibrium points of the Model 1 are obtained by solving the system of equations

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x_1 = r_1 x_1 \left(1 - \dfrac{e_1}{r_1}\right) - \dfrac{r_1 x_1^2}{K_1} + \dfrac{r_1 b_{12} x_1 x_2}{K_1} = 0, \\ \\ {}_{t_0}^{C}D_t^{\alpha} x_2 = r_2 x_2 \left(1 - \dfrac{e_2}{r_2}\right) - \dfrac{r_2 x_2^2}{K_2} + \dfrac{r_2 b_{21} x_1 x_2}{K_2} = 0. \end{cases} \tag{4.1.1}$$

We derive four equilibrium points as follows:
1. The origin $E_{10}(0,0)$, which represents extinction of both species.
2. $E_{11}(K_1A_1, 0)$, where $A_1 = 1 - \dfrac{e_1}{r_1}$, which represents extinction of the second species (x_2). The existence condition of E_{11} is $0 < e_1 < r_1$.
3. $E_{12}(0, K_2A_2)$, where $A_2 = 1 - \dfrac{e_2}{r_2}$, which represents extinction of the first species (x_1). The existence condition of E_{12} is $0 < e_2 < r_2$.
4. $E_{13}(x_1^*, x_2^*)$, where $x_1^* = \dfrac{A_1K_1 + b_{12}A_2K_2}{1 - b_{12}b_{21}}$, and $x_2^* = \dfrac{A_2K_2 + b_{21}A_1K_1}{1 - b_{12}b_{21}}$, which is called the coexistence equilibrium point.

As for the equilibrium points E_{10}, E_{11} and E_{12}, they are called the non-coexistence equilibrium points. The existence condition of E_{13} is presented in the following proposition.

Proposition 4.1.1. *If*

$$b_{12}b_{21} < 1, \ 0 < e_1 \leq r_1, \ \textit{and } 0 < e_2 < r_2 \tag{4.1.2}$$

or

$$b_{12}b_{21} < 1, \ 0 < e_1 < r_1, \ \textit{and } 0 < e_2 \leq r_2. \tag{4.1.3}$$

Then there exists a unique coexisting equilibrium $E_{13}(x_1^*, x_2^*)$ *of the fractional order system* (3.0.7).

Proof. The proof for this proposition is the same as that for Proposition 2 of reference [78], which is the proof that there exists a unique coexisting equilibrium point for the integer order system (3.0.1). □

4.1.1 Local stability of non-coexistence equilibrium points

In this subsection, we examine the local stability of the first three-quilibrium points of the system (3.0.7). The Jacobian matrix of the system is

$$J(x_1, x_2) = \begin{pmatrix} r_1 - e_1 - \dfrac{2r_1x_1}{K_1} + \dfrac{r_1b_{12}x_2}{K_1} & \dfrac{r_1b_{12}x_1}{K_1} \\ \dfrac{r_2b_{21}x_2}{K_2} & r_2 - e_2 - \dfrac{2r_2x_2}{K_2} + \dfrac{r_2b_{21}x_1}{K_2} \end{pmatrix}. \qquad (4.1.4)$$

Theorem 4.1.2. *If* $r_1 < e_1$ *and* $r_2 < e_2$, *then the equilibrium point* $E_{10}(0,0)$ *of the fractional order system* (3.0.7) *is locally asymptotically stable.*

Proof. From (4.1.4), the Jacobian matrix $J(E_{10})$ is given by:

$$J(E_{10}) = \begin{pmatrix} r_1 - e_1 & 0 \\ 0 & r_2 - e_2 \end{pmatrix}. \qquad (4.1.5)$$

Thus, the eigenvalues of $J(E_{10})$ are $\lambda_1 = r_1 - e_1$ and $\lambda_2 = r_2 - e_2$. By the assumptions of this theorem, it is easily seen that $\lambda_1 < 0$ and $\lambda_2 < 0$. Thus $|\arg\lambda_1| = |\arg\lambda_2| = \pi$. Therefore, according to Theorem 2.3.10, the equilibrium point $E_{10}(0,0)$ is locally asymptotically stable. □

Theorem 4.1.3. *If* $e_1 < r_1$ *and* $r_2 < \dfrac{e_2K_2}{K_2 + b_{21}K_1}$, *then the second species extinction equilibrium point* $E_{11}(K_1A_1, 0)$ *of the fractional order system* (3.0.7) *is locally asymptotically stable.*

Proof. By using (4.1.4), the Jacobian matrix $J(E_{11})$ can be obtained as follows:

$$J(E_{11}) = \begin{pmatrix} e_1 - r_1 & r_1b_{12}A_1 \\ 0 & r_2 - e_2 + \dfrac{r_2b_{21}K_1A_1}{K_2} \end{pmatrix}. \qquad (4.1.6)$$

Solving the characteristic equation: $det(\lambda I - J(E_{11})) = 0$ for λ to find the eigenvalues of $J(E_{11})$:

$$(\lambda - (e_1 - r_1))(\lambda - (r_2 - e_2 + \frac{r_2b_{21}K_1A_1}{K_2})) = 0.$$

So, the eigenvalues of $J(E_{11})$ are $\lambda_3 = e_1 - r_1$ and $\lambda_4 = r_2 - e_2 + \dfrac{r_2 b_{21} K_1 A_1}{K_2}$. By the assumption, it is clear that $\lambda_3 < 0$ and $\lambda_4 < 0$. According to Theorem 2.3.10, the equilibrium point $E_{11}(K_1 A_1, 0)$ is locally asymptotically stable. This completes the proof. $\square$

Theorem 4.1.4. *If $r_1 < \dfrac{e_1 K_1}{K_1 + b_{12} K_2}$ and $e_2 < r_2$, then the first species extinction equilibrium point $E_{12}(0, K_2 A_2)$ of the fractional order system* (3.0.7) *is locally asymptotically stable.*

Proof. By using (4.1.4), the Jacobian matrix $J(E_{12})$ can be obtained as follows:

$$J(E_{12}) = \begin{pmatrix} r_1 - e_1 + \dfrac{r_1 b_{12} K_2 A_2}{K_1} & 0 \\ r_2 b_{21} A_2 & e_2 - r_2 \end{pmatrix}. \tag{4.1.7}$$

Solving the characteristic equation: $det(\lambda I - J(E_{12})) = 0$ for λ to find the eigenvalues of $J(E_{11})$:

$$(\lambda - (r_1 - e_1 + \frac{r_1 b_{12} K_2 A_2}{K_1}))(\lambda - (e_2 - r_2)) = 0.$$

Hence, the eigenvalues of $J(E_{12})$ are $\lambda_5 = r_1 - e_1 + \dfrac{r_1 b_{12} K_2 A_2}{K_1}$ and $\lambda_6 = e_2 - r_2$. By the assumption, we have $\lambda_5 < 0$ and $\lambda_6 < 0$. From Theorem 2.3.10, the equilibrium point $E_{12}(0, K_2 A_2)$ is locally asymptotically stable. $\square$

4.1.2 Global stability of positive coexistence equilibrium

In this subsection, we investigate the sufficient conditions for the global uniform asymptotic stability of the positive coexisting equilibrium for the corresponding fractional order system (3.0.7) using the Lyapunov function.

Theorem 4.1.5. *If one of the following conditions holds:*

(i) $b_{12} b_{21} < 1,\ 0 < e_1 \le r_1,$ *and* $0 < e_2 < r_2$,

(ii) $b_{12} b_{21} < 1,\ 0 < e_1 < r_1,$ *and* $0 < e_2 \le r_2$.

Then, the unique interior positive equilibrium $E_{13}(x_1^*, x_2^*)$ of the fractional order system (3.0.7) *is globally uniformly asymptotically stable on $\mathbb{R}_+^2$.*

Proof. We define by $V_1 : \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 > 0,\ x_2 > 0\} \to \mathbb{R}$, such that

$$V_1(x_1, x_2) = c_1 \int_{x_1^*}^{x_1} \frac{\theta - x_1^*}{\theta} d\theta + c_2 \int_{x_2^*}^{x_2} \frac{\theta - x_2^*}{\theta} d\theta, \tag{4.1.8}$$

where $c_1 > 0$ and $c_2 = \dfrac{r_1 b_{12} K_2}{r_2 b_{21} K_1} c_1$. The above equation (4.1.8) can be written in the form

$$V_1(x_1, x_2) = c_1 x_1^* \left(\frac{x_1}{x_1^*} - 1 - \ln \frac{x_1}{x_1^*} \right) + c_2 x_2^* \left(\frac{x_2}{x_2^*} - 1 - \ln \frac{x_2}{x_2^*} \right). \tag{4.1.9}$$

The function $V_1(x_1, x_2)$ is defined, continuous on domain $\mathbb{R}_+^2$ and $V_1(x_1, x_2) > 0$ for all $(x_1, x_2) \in \mathbb{R}_+^2 \backslash (x_1^*, x_2^*)$, while $V_1(x_1, x_2) = 0$ if $x_1 = x_1^*$ and $x_2 = x_2^*$. Hence, by Definition 2.3.11, we get that $V_1(x_1, x_2)$ is a Lyapunov function. Also, $V_1(x_1, x_2)$ tends to $+\infty$ if either x_1 or x_2 tends to 0 or to $+\infty$. These properties mean that $V_1(x_1, x_2)$ is radially unbounded. According to Theorem 2.3.12, we can pick $W_1(x) = W_2(x) = V_1(x)$, thus the condition (2.3.4) holds. By applying the linearity property of the Caputo fractional derivative and using Theorem 2.3.13, we obtain

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha} V_1(x_1, x_2) &= {}_{t_0}^{C}D_t^{\alpha} \left(c_1 \left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) \right) + {}_{t_0}^{C}D_t^{\alpha} \left(c_2 \left(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right) \right) \\
&= c_1 \left({}_{t_0}^{C}D_t^{\alpha} \left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) \right) + c_2 \left({}_{t_0}^{C}D_t^{\alpha} \left(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right) \right) \\
&\leq c_1 \left(1 - \frac{x_1^*}{x_1} \right) {}_{t_0}^{C}D_t^{\alpha} x_1 + c_2 \left(1 - \frac{x_2^*}{x_2} \right) {}_{t_0}^{C}D_t^{\alpha} x_2 \\
&= c_1 \left(1 - \frac{x_1^*}{x_1} \right) \left(r_1 x_1 \left(A_1 - \frac{x_1}{K_1} \right) + \frac{r_1 b_{12}}{K_1} x_1 x_2 \right) \\
&\quad + c_2 \left(1 - \frac{x_2^*}{x_2} \right) \left(r_2 x_2 \left(A_2 - \frac{x_2}{K_2} \right) + \frac{r_2 b_{21}}{K_2} x_1 x_2 \right) \\
&= \frac{c_1 r_1}{K_1} (x_1 - x_1^*)(A_1 K_1 - x_1 + b_{12} x_2) + \frac{c_2 r_2}{K_2} (x_2 - x_2^*)(A_2 K_2 - x_2 + b_{21} x_1).
\end{aligned}$$

Using Proposition 4.1.1, the positive equilibrium point $E_{13}(x_1^*, x_2^*)$ of the system (3.0.7) satisfies the equalities

$$A_1 K_1 = x_1^* - b_{12} x_2^* \text{ and } A_2 K_2 = x_2^* - b_{21} x_1^*.$$

Consequently,

$$\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha} V_1(x_1, x_2) &\leq \frac{c_1 r_1}{K_1} (x_1 - x_1^*)(x_1^* - b_{12} x_2^* - x_1 + b_{12} x_2) \\
&\quad + \frac{c_2 r_2}{K_2} (x_2 - x_2^*)(x_2^* - b_{21} x_1^* - x_2 + b_{21} x_1) \\
&= \frac{c_1 r_1}{K_1} (x_1 - x_1^*)(-(x_1 - x_1^*) + b_{12}(x_2 - x_2^*)) \\
&\quad + \frac{c_2 r_2}{K_2} (x_2 - x_2^*)(-(x_2 - x_2^*) + b_{21}(x_1 - x_1^*)) \\
&= \frac{c_1 r_1}{K_1} (-(x_1 - x_1^*)^2 + b_{12}(x_1 - x_1^*)(x_2 - x_2^*)) \\
&\quad + \frac{c_2 r_2}{K_2} (-(x_2 - x_2^*)^2 + b_{21}(x_1 - x_1^*)(x_2 - x_2^*)).
\end{aligned}$$

Since $c_2 = \dfrac{r_1 b_{12} K_2}{r_2 b_{21} K_1} c_1$, we have

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha} V_1(x_1, x_2) &\leq \frac{c_1 r_1}{K_1}\Bigg(-(x_1 - x_1^*)^2 - \frac{b_{12}}{b_{21}}(x_2 - x_2^*)^2 \\
&\qquad +2b_{12}(x_1 - x_1^*)(x_2 - x_2^*)\Bigg) \\
&= -\frac{c_1 r_1}{K_1}\Bigg((x_1 - x_1^*)^2 - 2(x_1 - x_1^*)b_{12}(x_2 - x_2^*) + b_{12}^2(x_2 - x_2^*)^2 \\
&\qquad -b_{12}^2(x_2 - x_2^*)^2 + \frac{b_{12}}{b_{21}}(x_2 - x_2^*)^2\Bigg) \\
&= -\frac{c_1 r_1}{K_1}\Bigg(((x_1 - x_1^*) - b_{12}(x_2 - x_2^*))^2 + \frac{b_{12}}{b_{21}}(1 - b_{12}b_{21})(x_2 - x_2^*)^2\Bigg).
\end{aligned}
$$

Suppose that

$$
W_3(x) = \frac{c_1 r_1}{K_1}\Bigg(((x_1 - x_1^*) - b_{12}(x_2 - x_2^*))^2 + \frac{b_{12}}{b_{21}}(1 - b_{12}b_{21})(x_2 - x_2^*)^2\Bigg).
$$

It can be verified that $W_3(x)$ is defined, continuous on domain $\mathbb{R}_+^2$ and $W_3(x) > 0$ for all $x = (x_1, x_2) \in \mathbb{R}_+^2 \backslash (x_1^*, x_2^*)$, while $W_3(x) = 0$ if $x_1 = x_1^*$ and $x_2 = x_2^*$. So, $W_3(x)$ is a positive definite function. Thus, we obtain ${}_{t_0}^{C}D_t^{\alpha} V_1(t, x) \leq -W_3(x)$. By Theorem 2.3.12 and V_1 is radially unbounded, then the positive equilibrium point $E_{13}(x_1^*, x_2^*)$ of the system (3.0.7) is globally uniformly asymptotically stable on $\mathbb{R}_+^2$. □

Remark 4.1.6. *The Lyapunov function which is defined in Theorem 4.1.5 is the family of Volterra-type Lyapunov function that using in the integer-order differential Lotka-Volterra equations [78–81].*

4.2 Model 2

The equilibrium points of the Model 2 are obtained by solving the system of equations

$$
\begin{cases}
{}_{t_0}^{C}D_t^{\alpha} x_1 = r_1 x_1 \left(1 - \dfrac{e_1}{r_1}\right) - \dfrac{r_1 x_1^2}{K_1 + b_{12} x_2} = 0, \\[2ex]
{}_{t_0}^{C}D_t^{\alpha} x_2 = r_2 x_2 \left(1 - \dfrac{e_2}{r_2}\right) - \dfrac{r_2 x_2^2}{K_2 + b_{21} x_1} = 0.
\end{cases}
\tag{4.2.1}
$$

We obtain four equilibrium points as follows:

1. The origin $E_{20}(0, 0)$, which represents extinction of both species.
2. $E_{21}(K_1 A_1, 0)$, where $A_1 = 1 - \dfrac{e_1}{r_1}$, which represents extinction of the second species (x_2). The existence condition of E_{21} is $0 < e_1 < r_1$.
3. $E_{22}(0, K_2 A_2)$, where $A_2 = 1 - \dfrac{e_2}{r_2}$, which represents extinction of the first species (x_1).

The existence condition of E_{22} is $0 < e_2 < r_2$.

4. $E_{23}(x_1^*, x_2^*)$, where $x_1^* = \dfrac{A_1(K_1 + b_{12}A_2K_2)}{1 - A_1A_2b_{12}b_{21}}$, $x_2^* = \dfrac{A_2(K_2 + b_{21}A_1K_1)}{1 - A_1A_2b_{12}b_{21}}$, which is called the coexistence equilibrium point.

As for the equilibrium points E_{20}, E_{21} and E_{22}, they are called the non-coexistence equilibrium points. The existence condition of E_{23} is obtained in the next proposition.

Proposition 4.2.1. *If*

$$A_1A_2b_{12}b_{21} < 1,\ 0 < e_1 < r_1,\ \text{and } 0 < e_2 < r_2. \tag{4.2.2}$$

Then there exists a unique coexisting equilibrium $E_{23}(x_1^*, x_2^*)$ of the fractional order system (3.0.8).

Proof. The proof for this proposition is the same as that for Proposition 1 of reference [78], which is the proof that there exists a unique coexisting equilibrium point for the integer order system (3.0.2). $\square$

4.2.1 Local stability of non-coexistence equilibrium points

The local stability of the first three equilibrium points of the system (3.0.8) are analyzed. The Jacobian matrix of the system is

$$J(x_1, x_2) = \begin{pmatrix} r_1 - e_1 - \dfrac{2r_1x_1}{K_1 + b_{12}x_2} & \dfrac{b_{12}r_1x_1^2}{(K_1 + b_{12}x_2)^2} \\ \dfrac{b_{21}r_2x_2^2}{(K_2 + b_{21}x_1)^2} & r_2 - e_2 - \dfrac{2r_2x_2}{K_2 + b_{21}x_1} \end{pmatrix}. \tag{4.2.3}$$

Theorem 4.2.2. *If $r_1 < e_1$ and $r_2 < e_2$, then the equilibrium point $E_{20}(0,0)$ of the fractional order system* (3.0.8) *is locally asymptotically stable.*

Proof. From (4.2.3), the Jacobian matrix $J(E_{20})$ is given as follows:

$$J(E_{20}) = \begin{pmatrix} r_1 - e_1 & 0 \\ 0 & r_2 - e_2 \end{pmatrix}. \tag{4.2.4}$$

Therefore, the eigenvalues of $J(E_{20})$ are $\lambda_1 = r_1 - e_1$ and $\lambda_2 = r_2 - e_2$. By the assumptions of this theorem, $\lambda_1 < 0$ and $\lambda_2 < 0$. Thus $|\arg \lambda_1| = |\arg \lambda_2| = \pi$. Therefore, according to Theorem 2.3.10, the equilibrium point $E_{20}(0,0)$ is locally asymptotically stable. $\square$

Theorem 4.2.3. *If $e_1 < r_1$ and $r_2 < e_2$, then the second species extinction equilibrium point $E_{21}(K_1A_1, 0)$ of the fractional order system* (3.0.8) *is locally asymptotically stable.*

Proof. By using (4.2.3), the Jacobian matrix $J(E_{21})$ can be obtained as follows:

$$J(E_{21}) = \begin{pmatrix} e_1 - r_1 & b_{12} r_1 A_1^2 \\ 0 & r_2 - e_2 \end{pmatrix}. \tag{4.2.5}$$

Solving the characteristic equation: $det(\lambda I - J(E_{21})) = 0$ for λ to find the eigenvalues of $J(E_{21})$:

$$(\lambda - (e_1 - r_1))(\lambda - (r_2 - e_2)) = 0.$$

Thus, the eigenvalues of $J(E_{21})$ are $\lambda_3 = e_1 - r_1$ and $\lambda_4 = r_2 - e_2$. By the assumption, it is clear that $\lambda_3 < 0$ and $\lambda_4 < 0$. According to Theorem 2.3.10, the equilibrium point $E_{21}(K_1 A_1, 0)$ is locally asymptotically stable. This completes the proof. $\square$

Theorem 4.2.4. *If $r_1 < e_1$ and $e_2 < r_2$, then the first species extinction equilibrium point $E_{22}(0, K_2 A_2)$ of the fractional order system* (3.0.8) *is locally asymptotically stable.*

Proof. By using (4.2.3), the Jacobian matrix $J(E_{22})$ can be obtained as follows:

$$J(E_{22}) = \begin{pmatrix} r_1 - e_1 & 0 \\ b_{21} r_2 A_2^2 & e_2 - r_2 \end{pmatrix}. \tag{4.2.6}$$

Solving the characteristic equation: $det(\lambda I - J(E_{22})) = 0$ for λ to find the eigenvalues of $J(E_{22})$:

$$(\lambda - (r_1 - e_1))(\lambda - (e_2 - r_2)) = 0.$$

Therefore, the eigenvalues of $J(E_{22})$ are $\lambda_5 = r_1 - e_1$ and $\lambda_6 = e_2 - r_2$. By the above assumption, we obtain $\lambda_5 < 0$ and $\lambda_6 < 0$. From Theorem 2.3.10, we can conclude that the equilibrium point $E_{22}(0, K_2 A_2)$ is locally asymptotically stable. $\square$

4.2.2 Global stability of positive coexistence equilibrium

The sufficient conditions for the global uniform asymptotic stability of the positive coexisting equilibrium for the corresponding fractional order system using the Lyapunov function are investigated.

Theorem 4.2.5. *If $A_1 b_{12} < 2$, $A_2 b_{21} < 2$, $x_1 < \left(\frac{2 - A_2 b_{21}}{A_2 b_{21}}\right) x_2$, $x_2 < \left(\frac{2 - A_1 b_{12}}{A_1 b_{12}}\right) x_1$, $A_1 A_2 b_{12} b_{21} < 1$, $0 < e_1 < r_1$, and $0 < e_2 < r_2$. Then the unique interior positive equilibrium $E_{23}(x_1^*, x_2^*)$ of the fractional order system* (3.0.8) *is globally uniformly asymptotically stable on $\mathbb{R}_+^2$.*

Proof. We define by $V_2 : \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 > 0,\, x_2 > 0\} \to \mathbb{R}$, such that

$$V_2(x_1, x_2) = c_3 \int\limits_{x_1^*}^{x_1} \frac{\theta - x_1^*}{K_2 + b_{21}\theta} d\theta + c_4 \int\limits_{x_2^*}^{x_2} \frac{\theta - x_2^*}{K_1 + b_{12}\theta} d\theta, \tag{4.2.7}$$

where $c_3 > 0$ and $c_4 = \dfrac{r_1 b_{12} A_1}{r_2 b_{21} A_2} c_3$. So that (4.2.7) can be written in the following form

$$\begin{aligned} V_2(x_1, x_2) = {} & \frac{c_3}{b_{21}} \left(x_1 - x_1^* - \left(\frac{K_2 + b_{21}x_1^*}{b_{21}} \right) \ln \left(\frac{K_2 + b_{21}x_1}{K_2 + b_{21}x_1^*} \right) \right) \\ & + \frac{c_4}{b_{12}} \left(x_2 - x_2^* - \left(\frac{K_1 + b_{12}x_2^*}{b_{12}} \right) \ln \left(\frac{K_1 + b_{12}x_2}{K_1 + b_{12}x_2^*} \right) \right). \end{aligned} \tag{4.2.8}$$

The function $V_2(x_1, x_2)$ is defined, continuous on domain $\mathbb{R}^2_+$ and $V_2(x_1, x_2) > 0$ for all $(x_1, x_2) \in \mathbb{R}^2_+ \backslash (x_1^*, x_2^*)$, while $V_2(x_1, x_2) = 0$ if $x_1 = x_1^*$ and $x_2 = x_2^*$. Thus, we can see that $V_2(x_1, x_2)$ is a Lyapunov function according to Definition 2.3.11. Also, $V_2(x_1, x_2)$ tends to $+\infty$ if either x_1 or x_2 tends to 0 or to $+\infty$. These properties mean that $V_2(x_1, x_2)$ is radially unbounded. According to Theorem 2.3.12, we can pick $W_1(x) = W_2(x) = V_2(x)$, so the condition (2.3.4) holds. By applying the linearity property of the Caputo fractional derivative and using Theorem 2.3.13, we obtain

$$\begin{aligned} {}^{C}_{t_0}D^{\alpha}_{t} V_2(x_1, x_2) = {} & {}^{C}_{t_0}D^{\alpha}_{t} \left(\frac{c_3}{b_{21}} \left(x_1 - x_1^* - \left(\frac{K_2 + b_{21}x_1^*}{b_{21}} \right) \ln \left(\frac{K_2 + b_{21}x_1}{K_2 + b_{21}x_1^*} \right) \right) \right) \\ & + {}^{C}_{t_0}D^{\alpha}_{t} \left(\frac{c_4}{b_{12}} \left(x_2 - x_2^* - \left(\frac{K_1 + b_{12}x_2^*}{b_{12}} \right) \ln \left(\frac{K_1 + b_{12}x_2}{K_1 + b_{12}x_2^*} \right) \right) \right) \\ \le {} & \frac{c_3}{b_{21}} \left({}^{C}_{t_0}D^{\alpha}_{t} x_1 - \left(\frac{K_2 + b_{21}x_1^*}{K_2 + b_{21}x_1} \right) {}^{C}_{t_0}D^{\alpha}_{t} x_1 \right) \\ & + \frac{c_4}{b_{12}} \left({}^{C}_{t_0}D^{\alpha}_{t} x_2 - \left(\frac{K_1 + b_{12}x_2^*}{K_1 + b_{12}x_2} \right) {}^{C}_{t_0}D^{\alpha}_{t} x_2 \right) \\ = {} & \frac{c_3}{b_{21}} \left(1 - \frac{K_2 + b_{21}x_1^*}{K_2 + b_{21}x_1} \right) {}^{C}_{t_0}D^{\alpha}_{t} x_1 + \frac{c_4}{b_{12}} \left(1 - \frac{K_1 + b_{12}x_2^*}{K_1 + b_{12}x_2} \right) {}^{C}_{t_0}D^{\alpha}_{t} x_2 \\ = {} & \frac{c_3}{K_2 + b_{21}x_1} (x_1 - x_1^*) \, {}^{C}_{t_0}D^{\alpha}_{t} x_1 + \frac{c_4}{K_1 + b_{12}x_2} (x_2 - x_2^*) \, {}^{C}_{t_0}D^{\alpha}_{t} x_2 \\ = {} & \frac{c_3}{K_2 + b_{21}x_1} (x_1 - x_1^*) \left(r_1 x_1 \left(1 - \frac{e_1}{r_1} \right) - \frac{r_1 x_1^2}{K_1 + b_{12}x_2} \right) \\ & + \frac{c_4}{K_1 + b_{12}x_2} (x_2 - x_2^*) \left(r_2 x_2 \left(1 - \frac{e_2}{r_2} \right) - \frac{r_2 x_2^2}{K_2 + b_{21}x_1} \right) \\ = {} & \frac{c_3 r_1 x_1}{(K_1 + b_{12}x_2)(K_2 + b_{21}x_1)} (x_1 - x_1^*) (A_1 K_1 + A_1 b_{12} x_2 - x_1) \\ & + \frac{c_4 r_2 x_2}{(K_1 + b_{12}x_2)(K_2 + b_{21}x_1)} (x_2 - x_2^*) (A_2 K_2 + A_2 b_{21} x_1 - x_2). \end{aligned}$$

Using Proposition 4.2.1, the positive equilibrium point (x_1^*, x_2^*) of the system (3.0.8) satisfies the equalities

$$A_1 K_1 = x_1^* - A_1 b_{12} x_2^* \text{ and } A_2 K_2 = x_2^* - A_2 b_{21} x_1^*.$$

It is seen that

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V_2(x_1,x_2) \leq{}& \frac{c_3 r_1 x_1}{(K_1+b_{12}x_2)(K_2+b_{21}x_1)}(x_1-x_1^*)\left(-(x_1-x_1^*)+A_1b_{12}(x_2-x_2^*)\right)\\
&+\frac{c_4 r_2 x_2}{(K_1+b_{12}x_2)(K_2+b_{21}x_1)}(x_2-x_2^*)\left(-(x_2-x_2^*)+A_2b_{21}(x_1-x_1^*)\right).
\end{aligned}
$$

Since $c_4=\dfrac{r_1b_{12}A_1}{r_2b_{21}A_2}c_3$, we have

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\alpha}V_2(x_1,x_2) \leq{}& \frac{c_3 r_1}{(K_1+b_{12}x_2)(K_2+b_{21}x_1)}\Bigg(-x_1(x_1-x_1^*)^2-x_2\frac{A_1b_{12}}{A_2b_{21}}(x_2-x_2^*)^2\\
&+A_1b_{12}(x_1+x_2)(x_1-x_1^*)(x_2-x_2^*)\Bigg)\\
\leq{}& \frac{c_3 r_1}{(K_1+b_{12}x_2)(K_2+b_{21}x_1)}\Bigg(-x_1(x_1-x_1^*)^2-x_2\frac{A_1b_{12}}{A_2b_{21}}(x_2-x_2^*)^2\\
&+\frac{1}{2}A_1b_{12}(x_1+x_2)(x_1-x_1^*)^2+\frac{1}{2}A_1b_{12}(x_1+x_2)(x_2-x_2^*)^2\Bigg)\\
={}& \frac{-c_3 r_1}{(K_1+b_{12}x_2)(K_2+b_{21}x_1)}\Bigg(\left(x_1-\frac{1}{2}A_1b_{12}(x_1+x_2)\right)(x_1-x_1^*)^2\\
&+\left(x_2\frac{A_1b_{12}}{A_2b_{21}}-\frac{1}{2}A_1b_{12}(x_1+x_2)\right)(x_2-x_2^*)^2\Bigg).
\end{aligned}
$$

Suppose that

$$
\begin{aligned}
W_3(x)={}& \frac{c_3 r_1}{(K_1+b_{12}x_2)(K_2+b_{21}x_1)}\Bigg(\left(x_1-\frac{1}{2}A_1b_{12}(x_1+x_2)\right)(x_1-x_1^*)^2\\
&+\left(x_2\frac{A_1b_{12}}{A_2b_{21}}-\frac{1}{2}A_1b_{12}(x_1+x_2)\right)(x_2-x_2^*)^2\Bigg).
\end{aligned}
$$

It can be verified that $W_3(x)$ is defined, continuous on domain $\mathbb{R}_+^2$ and $W_3(x)>0$ for all $x=(x_1,x_2)\in\mathbb{R}_+^2\backslash(x_1^*,x_2^*)$, while $W_2(x)=0$ if $x_1=x_1^*$ and $x_2=x_2^*$. So, $W_3(x)$ is a positive definite function. Thus, ${}_{t_0}^{C}D_t^{\alpha}V_2(t,x)\leq -W_3(x)$. By Theorem 2.3.12 and V_2 is radially unbounded, we can conclude that the equilibrium point $E_{23}(x_1^*,x_2^*)$ of the system (3.0.8) is globally uniformly asymptotically stable on $\mathbb{R}_+^2$. □

Remark 4.2.6. *The Lyapunov function in Theorem 4.2.5 is modified from the function of integer-order differential systems presented in [82] and different from the functions presented in [57–60].*

Remark 4.2.7. *Theorems 4.1.5 and 4.2.5 can be applied to study a facultative mutualism of two species. In particular, the interaction between two species is assumed to be described by the models* (3.0.7) *and* (3.0.8) *where the parameters satisfy the conditions in the theorem. Subsequently, any solutions starting at a positive initial point eventually tend to the positive coexistence equilibrium of the model. This means biologically that the two species always coexist in the same habitat.*