CHAPTER 6

Conclusion

In this thesis we presented two Caputo fractional order models of two-species facultative mutualism with harvesting and analyzed the stability of the proposed models. By stability analysis, we obtain the sufficient conditions on the parameter for the local asymptotic stability of the three non-coexistence equilibrium points using the linearization method and the global uniform asymptotic stability of the coexistence equilibrium point via the Lyapunov's direct method.

For the Model 1, it has four equilibrium points, which are the non-coexistence equilibrium points $E_{10}(0,0),\ E_{11}(K_1A_1,0),\ E_{12}(0,K_2A_2)$ and the coexistence positive equilibrium point $E_{13}(x_1^*,x_2^*)$.

The sufficient conditions of the locally asymptotically stable of the non-coexistence equilibrium points $E_{10},\ E_{11}$ and E_{12} are provided in Theorems 4.1.2, 4.1.3 and 4.1.4, respectively. The equilibrium point E_{10} is locally asymptotically stable if the model parameters satisfies $r_1 < e_1$ and $r_2 < e_2$, while E_{11} is locally asymptotically stable if $e_1 < r_1$ and $r_2 < \dfrac{e_2K_2}{K_2 + b_{21}K_1}$. In addition, E_{12} is locally asymptotically stable if $r_1 < \dfrac{e_1K_1}{K_1 + b_{12}K_2}$ and $e_2 < r_2$.

For the unique coexistence equilibrium point E_{13}, we also obtain the conditions for globally uniformly asymptotically stable, which appear in Theorem 4.1.5, that is, $b_{12}b_{21} < 1,\ 0 < e_1 \leq r_1,\ 0 < e_2 < r_2$ or $b_{12}b_{21} < 1,\ 0 < e_1 < r_1,\ 0 < e_2 \leq r_2$.

Likewise, the Model 2 also has four equilibrium points including the non-coexistence equilibrium points $E_{20}(0,0),\ E_{21}(K_1A_1,0),\ E_{22}(0,K_2A_2)$ and the coexistence positive equilibrium point $E_{23}(x_1^*,x_2^*)$. We reveal the sufficient conditions for the locally asymptotically stable of the non-coexistence equilibrium points $E_{20},\ E_{21}$ and E_{22} in Theorems 4.2.2, 4.2.3 and 4.2.4, respectively. The local asymptotic stability of the trivial equilibrium point E_{20} is obtained if the model parameters satisfies $r_1 < e_1$ and $r_2 < e_2$, which is the same conditions as the equilibrium point E_{10} of the Model 1, while E_{21} is locally asymptotically stable if $e_1 < r_1$ and $r_2 < e_2$ and E_{22} if $r_1 < e_1$ and $e_2 < r_2$.

The sufficient conditions of the global uniform asymptotic stability for the unique coexistence equilibrium point E_{23} is established in Theorem 4.2.5, that is, $A_1b_{12} < 2,\ A_2b_{21} < 2,\ x_1 < \left(\dfrac{2 - A_2b_{21}}{A_2b_{21}}\right)x_2,\ x_2 < \left(\dfrac{2 - A_1b_{12}}{A_1b_{12}}\right)x_1,\ A_1A_2b_{12}b_{21} < 1,\ 0 < e_1 < r_1$, and

$0 < e_2 < r_2$.

Finally, the numerical simulations of the Model 1 and Model 2 were presented using the given parameters in Tables 5.1 and 5.2, respectively. The parameters satisfy the sufficient conditions for asymptotic stability of each equilibrium point. From Figures 5.1 - 5.16, all solutions of the two models converged to the equilibrium points. In conclusion, the above results corresponded with the theoretical results of this study.