CHAPTER 1

Introduction

In mathematical finance, the Black-Scholes model [7] is a classical model for describing derivatives of the stock price with geometric Brownian motion behaviour. In other words, the stock price process $\{S_t\}_{t\geq 0}$ satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1.0.1}$$

where $\{B_t\}_{t\geq 0}$ is a standard Wiener process. The constant parameters μ and σ denote the drift and volatility of the process, respectively. The analytic solution of (1.0.1) is given by

$$S_t = S_0 \exp((\mu - \sigma^2/2)t + \sigma B_t).$$

Such model plays key roles in both research and practice.

The model assumes that the underlying volatility is a constant over the life of the derivative, and unaffected by the changes in the price level of the underlying security. However, the model cannot explain long-observed features of the implied volatility surface such as volatility smile, which indicates that implied volatility does tend to vary with respect to other factors such as strike price and expiry time. Stochastic volatility models are possible approaches to resolve the constant volatility problem by assuming that volatility of the underlying price is another stochastic process rather than a constant.

Let $\{v_t\}_{t\geq 0}$ be the volatility process. One of the first stochastic volatility models was introduced by Hull and White in 1987 [16]. The general form of this model is:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dB_t,$$
$$dv_t = \psi v_t dt + \xi v_t dW_t,$$

where ψ and ξ are mean and variance of the volatility process, respectively. Also, $\{B_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ represent two standard Wiener processes, possibly correlated with a correlation coefficient ρ. For this model, the correlation coefficient ρ is assumed to be zero.

In 1991, M. Stein and C. Stein [24] introduced a model with the volatility process being modeled as an Ornstein-Uhlenbeck process. The general form of this model is as follows:

$$
\begin{aligned}
dS_t &= \mu S_t dt + v_t S_t dB_t, \\
dv_t &= \kappa(\theta - v_t)dt + \xi dW_t,
\end{aligned}
$$

where θ is the mean long-term volatility, κ is the speed of mean-reversion and the correlation coefficient ρ is assumed as zero.

In 1993, Heston [14] introduced a model setting by the Cox-Ingersoll-Ross(CIR) process to define the volatility process. The general form of this model is:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dB_t, \\
dv_t &= \kappa(\theta - v_t)dt + \xi \sqrt{v_t} dW_t.
\end{aligned}
$$

This Heston model is commomly used as a stochastic volatility model.

In 1997, Heston [13] introduced a new model, the so-called 3/2 model. The general form of this model is:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dB_t, \\
dv_t &= \kappa v_t(\theta - v_t)dt + \xi v_t^{\frac{3}{2}} dW_t.
\end{aligned}
$$

Note that the volatility process of the 3/2 model is a mean-reversion process to the equilibrium point θ, and the speed of mean-reversion is κv_t. This means that the reversion also depends on the variance value itself. With reference to the real data, it means that the model can incorporate and explain fast volatility increases and decreases. Consequently, the model exhibits better agreement with empirical studies as compared to the original Heston model, [6, 17]. Additionally, applications of the 3/2 model for derivative pricing can be found in [8, 10].

Generally, obtaining an estimation of volatility from data can be difficult and unclear. This is due to the fact that the volatility process is a hidden process which can not be directly observed from the real data. This causes a filtering problem, so that it can only be approximated via the information of the underlying stock process. Some techniques such as maximum likelihood and particle filters were introduced to solve this problem, especially in the Heston model.

In 2007, Aït-Sahalia and Kimmel [2] developed and implemented a method for maximum likelihood estimation in a closed form of stochastic volatility models, in particular

CEV, Heston and GARCH models. They compared a full likelihood to an approximate likelihood procedure and the results in a small loss of accuracy related to the standard errors from a sampling noise.

In 2009, Atiya and Wall [5] proposed an analytic approximation for the likelihood function for volatility of the Heston model. Moreover, extension to the problem with fixed parameter estimation was also presented.

In 2009, Aihara, Bagchi and Saha [1] derived the optimal importance function and constructed the particle filter algorithm for the Heston model. Parameters contained in the model were also estimated by constructing the augmented states. The proposed method was appiled to the real data of the AEX index.

Motivated by those preceding studies combining with the idea given by Aihara *et al.* [1], we aimed to estimate the volatility process of the 3/2 volatility model by using the particle filter method. Moreover, we then used this estimated volatility process to estimate the model parameters by using the maximum likelihood estimation method.

The rest of this thesis is organized as follows : Chapter 2 is devoted to some useful definitions, notations and properties that were used throughout this thesis; Chapter 3 is for estimation of the volatility process and model parameters; finally, the conclusions of this thesis is given in Chapter 4.