# CHAPTER 3

## Main Results

In this chapter, we apply the SIS particle filter method and the maximum likelihood estimation method to estimate volatility process and model parameters in 3/2 model.

## 3.1 Estimation of Volatility Process

In this section, we derive how to use SIS particle filter method to estimate volatility process of 3/2 model. Section 3.1.1 provides the set up and the algorithm. The simulation study is in section 3.1.2.

### 3.1.1 Derivation of Particle Filter Algorithm

First, we recall the 3/2 stochastic volatility model can be described in the stochastic differential form as

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dB_t, \tag{3.1.1}$$

$$dv_t = \kappa v_t(\theta - v_t)dt + \xi v_t^{\frac{3}{2}} dW_t, \tag{3.1.2}$$

where $\{B_t\}_{t\geq 0}$ and $\{W_t\}_{t\geq 0}$ represent two standard Wiener processes, correlated with the correlation coefficient $\rho$.

Now, we define the log-price process

$$y_t := \log \frac{S_t}{S_0}.$$

And, we transform the system of SDEs in (3.1.1)-(3.1.2) to the following system:

**Lemma 3.1.1.** *The stochastic differential form of 3/2 volatility model in (3.1.1)-(3.1.2) can be written as*

$$dy_t = (\mu - \frac{1}{2}v_t)dt + \sqrt{v_t} dB_t, \tag{3.1.3}$$

$$dv_t = \big(\kappa v_t(\theta - v_t) - \xi\rho(\mu - \frac{1}{2}v_t)v_t\big)dt + \xi\rho v_t dy_t + \xi\sqrt{1-\rho^2}\, v_t^{\frac{3}{2}} dZ_t, \tag{3.1.4}$$

*where $\{B_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$ are independent standard Wiener processes.*

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*Proof.* To get (3.1.3), we use Ito's formula in Theorem 2.2.5 with (3.1.1) .
Since $F(t, S_t) = y_t$, $f(t, S_t) = \mu S_t$ and $g(t, S_t) = \sqrt{v_t}S_t$, we obtain that

$$\begin{aligned} dy_t &= \Big(\frac{\partial y_t}{\partial t} + \mu S_t \frac{\partial y_t}{\partial S_t} + \frac{1}{2}v_t S_t^2 \frac{\partial^2 y_t}{\partial S_t^2}\Big)dt + \sqrt{v_t}S_t\frac{\partial y_t}{\partial S_t}dB_t \\ &= \Big(0 + \mu S_t\frac{1}{S_t} - \frac{1}{2}v_t S_t^2\frac{1}{S_t^2}\Big)dt + \sqrt{v_t}S_t\frac{1}{S_t}dB_t \\ &= (\mu - \frac{1}{2}v_t)dt + \sqrt{v_t}dB_t. \end{aligned}$$

For (3.1.4), we know that $B_t$ and $W_t$ are two standard Wiener processes with the correlation coefficient $\rho$.
By setting $Z_t = (W_t - \rho B_t)/\sqrt{1-\rho^2}$, we get $Z_t$ is independent of $B_t$ (see Appendix A).
Moreover, by (3.1.3), we can write

$$\begin{aligned} dW_t &= \sqrt{1-\rho^2}dZ_t + \rho dB_t \\ &= \sqrt{1-\rho^2}dZ_t + \frac{\rho}{\sqrt{v_t}}\big(dy_t - (\mu - \frac{1}{2}v_t)dt\big). \end{aligned} \tag{3.1.5}$$

Therefore, by (3.1.2) and (3.1.5), we have

$$\begin{aligned} dv_t &= \kappa v_t(\theta - v_t)dt + \xi v_t^{\frac{3}{2}}\left(\sqrt{1-\rho^2}dZ_t + \frac{\rho}{\sqrt{v_t}}\big(dy_t - (\mu - \frac{1}{2}v_t)dt\big)\right) \\ &= \kappa v_t(\theta - v_t)dt + \xi\sqrt{1-\rho^2}v_t^{\frac{3}{2}}dZ_t + \xi\rho v_t\big(dy_t - (\mu - \frac{1}{2}v_t)dt\big) \\ &= \big(\kappa v_t(\theta - v_t) - \xi\rho(\mu - \frac{1}{2}v_t)v_t\big)dt + \xi\rho v_t dy_t + \xi\sqrt{1-\rho^2}v_t^{\frac{3}{2}}dZ_t. \end{aligned}$$

□

Note that, the purpose of transformation is to get the process $\{y_t\}_{t\geq 0}$ as a measurement of the volatility process $\{v_t\}_{t\geq 0}$. Next, we will apply the Euler-Maruyama method to (3.1.3)-(3.1.4). This leads to the following discretized system:

**Lemma 3.1.2.** *The Euler-Maruyama form of (3.1.3)-(3.1.4) are*

$$\widetilde{y}_{t_n} = \widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}_{t_n})\Delta t_n + \sqrt{\widetilde{v}_{t_{n-1}}}\Delta B_{t_n}, \tag{3.1.6}$$

$$\begin{aligned} \widetilde{v}_{t_n} = \widetilde{v}_{t_{n-1}} &+ \big(\kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}}) - \xi\rho(\mu - \frac{1}{2}\widetilde{v}_{t_{n-1}})\widetilde{v}_{t_{n-1}}\big)\Delta t_n \\ &+ \xi\rho\widetilde{v}_{t_{n-1}}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}) + \xi\sqrt{1-\rho^2}\widetilde{v}_{t_{n-1}}^{\frac{3}{2}}\Delta Z_{t_n}, \end{aligned} \tag{3.1.7}$$

*where* $\Delta t_n = t_n - t_{n-1}$, $\Delta B_{t_n} = B_{t_n} - B_{t_{n-1}}$ *and* $\Delta Z_{t_n} = Z_{t_n} - Z_{t_{n-1}}$.
*Moreover, an alternative form of the discretized volatility process in (3.1.7) is*

$$\widetilde{v}_{t_n} = \frac{\widetilde{v}_{t_{n-1}} + \kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}})\Delta t_n + \xi\widetilde{v}_{t_{n-1}}^{\frac{3}{2}}\Delta W_{t_n} + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}^2\Delta t_n}{1 + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}\Delta t_n}, \tag{3.1.8}$$

*where* $\Delta t_n = t_n - t_{n-1}$ *and* $\Delta W_{t_n} = W_{t_n} - W_{t_{n-1}}$.

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*Proof.* For (3.1.6) and (3.1.7), we use directly Euler-Maruyama method in Proposition 2.2.6. Then, for (3.1.8), from (3.1.6) and (3.1.7), we can see that

$$\begin{aligned}
\widetilde{v}_{t_n} + \frac{1}{2}\xi\rho\widetilde{v}_{t_n}\widetilde{v}_{t_{n-1}}\Delta t_n &= \widetilde{v}_{t_{n-1}} + \big(\kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}}) - \xi\rho(\mu - \frac{1}{2}\widetilde{v}_{t_{n-1}})\widetilde{v}_{t_{n-1}}\big)\Delta t_n \\
&\quad + \xi\sqrt{1-\rho^2}\widetilde{v}^{\frac{3}{2}}_{t_{n-1}}\Delta Z_{t_n} + \xi\rho\widetilde{v}_{t_{n-1}}\big((\mu - \frac{1}{2}\widetilde{v}_{t_n})\Delta t_n + \sqrt{\widetilde{v}_{t_{n-1}}}\Delta B_{t_n}\big) \\
&\quad + \frac{1}{2}\xi\rho\widetilde{v}_{t_n}\widetilde{v}_{t_{n-1}}\Delta t_n \\
&= \widetilde{v}_{t_{n-1}} + \kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}})\Delta t_n + \xi\sqrt{1-\rho^2}\widetilde{v}^{\frac{3}{2}}_{t_{n-1}}\Delta Z_{t_n} + \xi\rho\widetilde{v}^{\frac{3}{2}}_{t_{n-1}}\Delta B_{t_n} \\
&\quad + \frac{1}{2}\xi\rho\widetilde{v}^2_{t_{n-1}}\Delta t_n \\
&= \widetilde{v}_{t_{n-1}} + \kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}})\Delta t_n + \xi\widetilde{v}^{\frac{3}{2}}_{t_{n-1}}\Delta W_{t_n} + \frac{1}{2}\xi\rho\widetilde{v}^2_{t_{n-1}}\Delta t_n.
\end{aligned}$$

Since

$$\widetilde{v}_{t_n} + \frac{1}{2}\xi\rho\widetilde{v}_{t_n}\widetilde{v}_{t_{n-1}}\Delta t_n = \widetilde{v}_{t_n}\big(1 + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}\Delta t_n\big),$$

the proof is now completed. □

Now, we are ready to derive the SIS particle filter algorithm. Note that the update step is done via (3.1.7). For the analysis step, we consider the updated importance weight in order to estimate the discritized volatility process $\widetilde{v}_{t_n}$, for time $t_n$, based on our observation data $\{\widetilde{y}_{t_k}\}_{t_0 \le t_k \le t_n}$. As explained in section 2.3.1, our chosen updated importance weight $w^{(i)}_{t_n}$ at time $t_n$ is obtain as:

$$w^{(i)}_{t_n} \propto w^{(i)}_{t_{n-1}} \frac{p(\widetilde{y}_{t_n} \mid \widetilde{v}^{(i)}_{t_n})p(\widetilde{v}^{(i)}_{t_n} \mid \widetilde{v}^{(i)}_{t_{n-1}})}{p(\widetilde{v}^{(i)}_{t_n} \mid \widetilde{v}^{(i)}_{t_{n-1}}, \widetilde{y}_{t_n})}. \qquad (3.1.9)$$

Now, we derive all required probability density functions in (3.1.9).

**Proposition 3.1.3.** *Let $\phi(x,\mu,\sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$ be the probaility density function of $\mathcal{N}(\mu,\sigma^2)$ random variables. Then, from (3.1.9) and for $i = 1, 2, ..., N$,*

*(a) For the likelihood function,*

$$p(\widetilde{y}_{t_n} \mid \widetilde{v}^{(i)}_{t_n}) = \phi(\widetilde{y}_{t_n}, m_1, \sigma_1^2),$$

*where,*

$$m_1 = \widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}^{(i)}_{t_n})\Delta t_n, \text{ and } \ \sigma_1^2 = \widetilde{v}^{(i)}_{t_{n-1}}\Delta t_n.$$

*(b) For the transition probability density function,*

$$p(\widetilde{v}^{(i)}_{t_n} \mid \widetilde{v}^{(i)}_{t_{n-1}}) = \phi(\widetilde{v}^{(i)}_{t_n}, m_2, \sigma_2^2),$$

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*where*

$$m_2 = \frac{\widetilde{v}^{(i)}_{t_{n-1}} + \kappa \widetilde{v}^{(i)}_{t_{n-1}}(\theta - \widetilde{v}^{(i)}_{t_{n-1}})\Delta t_n + \frac{1}{2}\xi\rho(\widetilde{v}^{(i)}_{t_{n-1}})^2 \Delta t_n}{1 + \frac{1}{2}\xi\rho\widetilde{v}^{(i)}_{t_{n-1}}\Delta t_n},$$

*and*

$$\sigma_2^2 = \frac{\xi^2(\widetilde{v}^{(i)}_{t_{n-1}})^3 \Delta t_n}{(1 + \frac{1}{2}\xi\rho\widetilde{v}^{(i)}_{t_{n-1}}\Delta t_n)^2}.$$

(c) *For the optimal importance function,*

$$p(\widetilde{v}^{(i)}_{t_n} \mid \widetilde{v}^{(i)}_{t_{n-1}}, \widetilde{y}_{t_n}) = \phi(\widetilde{v}^{(i)}_{t_n}, m_3, \sigma_3^2),$$

*where*

$$m_3 = \widetilde{v}^{(i)}_{t_{n-1}} + \big(\kappa \widetilde{v}^{(i)}_{t_{n-1}}(\theta - \widetilde{v}^{(i)}_{t_{n-1}}) - \xi\rho(\mu - \frac{1}{2}\widetilde{v}^{(i)}_{t_{n-1}})\widetilde{v}^{(i)}_{t_{n-1}}\big)\Delta t_n + \xi\rho\widetilde{v}^{(i)}_{t_{n-1}}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}),$$

*and*

$$\sigma_3^2 = \xi^2(1-\rho^2)(\widetilde{v}^{(i)}_{t_{n-1}})^3 \Delta t_n.$$

*Proof.* We apply the properties of standard Wiener process and the properties of normal random variables to (3.1.6), (3.1.8) and (3.1.7), we obtain that $\widetilde{y}_{t_n}$ and $\widetilde{v}_{t_n}$ are also normal random variables for each time $t_n$ with mean and variance as follow:

(a) By properties of conditional expect value and conditional variance, we get

$$\begin{aligned}
m_1 &= \mathbb{E}(\widetilde{y}_{t_n} | \widetilde{v}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \mathbb{E}(\widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}_{t_n})\Delta t_n + \sqrt{\widetilde{v}_{t_{n-1}}}\Delta B_{t_n} | \widetilde{v}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}_{t_n})\Delta t_n + \sqrt{\widetilde{v}_{t_{n-1}}}\ \mathbb{E}(\Delta B_{t_n}) \\
&= \widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}_{t_n})\Delta t_n,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_1^2 &= \mathrm{var}(\widetilde{y}_{t_n} | \widetilde{v}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \mathrm{var}(\widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}_{t_n})\Delta t_n + \sqrt{\widetilde{v}_{t_{n-1}}}\Delta B_{t_n} | \widetilde{v}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \mathrm{var}(\sqrt{\widetilde{v}_{t_{n-1}}}\Delta B_{t_n} | \widetilde{v}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \widetilde{v}_{t_{n-1}}\mathrm{var}(\Delta B_{t_n}) \\
&= \widetilde{v}_{t_{n-1}}\Delta t_n.
\end{aligned}$$

For each $i = 1, 2, ..., N$, the proof is complete.

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(b) The idea of the proof for (b) is the same as the proof in (a) as:

$$
\begin{aligned}
m_2 &= \mathbb{E}(\widetilde{v}_{t_n}|\widetilde{v}_{t_{n-1}}) \\
&= \mathbb{E}\left(\frac{\widetilde{v}_{t_{n-1}} + \kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}})\Delta t_n + \xi\widetilde{v}_{t_{n-1}}^{\frac{3}{2}}\Delta W_{t_n} + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}^2\Delta t_n}{1 + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}\Delta t_n}\middle|\widetilde{v}_{t_{n-1}}\right) \\
&= \frac{\widetilde{v}_{t_{n-1}} + \kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}})\Delta t_n + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}^2\Delta t_n}{1 + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}\Delta t_n},
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_2^2 &= \mathrm{var}(\widetilde{v}_{t_n}|\widetilde{v}_{t_{n-1}}) \\
&= \mathrm{var}\left(\frac{\widetilde{v}_{t_{n-1}} + \kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}})\Delta t_n + \xi\widetilde{v}_{t_{n-1}}^{\frac{3}{2}}\Delta W_{t_n} + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}^2\Delta t_n}{1 + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}\Delta t_n}\middle|\widetilde{v}_{t_{n-1}}\right) \\
&= \frac{\xi^2\widetilde{v}_{t_{n-1}}^3\Delta t_n}{(1 + \frac{1}{2}\xi\rho\widetilde{v}_{t_{n-1}}\Delta t_n)^2}.
\end{aligned}
$$

For each $i = 1, 2, ..., N$, the proof is complete.

(c) Similarly with (a) and (b), we can see that

$$
\begin{aligned}
m_3 &= \mathbb{E}(\widetilde{v}_{t_n}|\widetilde{y}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \mathbb{E}\Big(\widetilde{v}_{t_{n-1}} + \big(\kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}}) - \xi\rho(\mu - \frac{1}{2}\widetilde{v}_{t_{n-1}})\widetilde{v}_{t_{n-1}}\big)\Delta t_n \\
&\qquad + \xi\rho\widetilde{v}_{t_{n-1}}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}) + \xi\sqrt{1-\rho^2}\widetilde{v}_{t_{n-1}}^{\frac{3}{2}}\Delta Z_{t_n}\big|\widetilde{y}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}\Big) \\
&= \widetilde{v}_{t_{n-1}} + \big(\kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}}) - \xi\rho(\mu - \frac{1}{2}\widetilde{v}_{t_{n-1}})\widetilde{v}_{t_{n-1}}\big)\Delta t_n \\
&\quad + \xi\rho\widetilde{v}_{t_{n-1}}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}),
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_3^2 &= \mathrm{var}(\widetilde{v}_{t_n}|\widetilde{y}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \mathrm{var}\Big(\widetilde{v}_{t_{n-1}} + \big(\kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}}) - \xi\rho(\mu - \frac{1}{2}\widetilde{v}_{t_{n-1}})\widetilde{v}_{t_{n-1}}\big)\Delta t_n \\
&\qquad + \xi\rho\widetilde{v}_{t_{n-1}}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}) + \xi\sqrt{1-\rho^2}\widetilde{v}_{t_{n-1}}^{\frac{3}{2}}\Delta Z_{t_n}\big|\widetilde{y}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}\Big) \\
&= \xi^2(1-\rho^2)\widetilde{v}_{t_{n-1}}^3\Delta t_n.
\end{aligned}
$$

For each $i = 1, 2, ..., N$, the proof is complete.

□

Combining all of these results, the SIS particle filter algorithm is now ready.

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**The SIS Particle Filter Algorithm for Volatility Process**

**Step 1:** For time step $t_0 = 0$ and for $i = 1, 2, ..., N$, draw the samples $\widetilde{v}_0^{(i)}$ from a prior distribution $p(\widetilde{v}_0)$ and set $w_0^{(i)} = \frac{1}{N}$.

**Step 2:** For time steps $t_n = 1, 2, 3, ..., M$,

**Step 2.1:** For $i = 1, 2, ..., N$, draw the samples $\widetilde{v}_{t_n}^{(i)}$ from (3.1.7).

**Step 2.2:** For $i = 1, 2, ..., N$, calculate the important weights $w_{t_n}^{(i)}$ according to (3.1.9) and normalize.

**Step 2.3:** Compute $\widetilde{N}_{eff}$ according to (2.3.8). If $\widetilde{N}_{eff} < N_T$, generate the new particle set by systematic resampling.

**Step 2.4:** Calculate the posterior distribution $p(\widetilde{v}_{t_n}|\widetilde{y}_{1:t_n})$ according to (2.3.5).

### 3.1.2 Simulation Studies of Volatility Estimation

In the simulation studies, we first create one realization of $S_t$ and $v_t$ based on the 3/2 stochastic volatility model for $0 \leq t \leq 5$, the time difference $\Delta t_n = 0.00025$ with parameter values $\mu = 0.1$, $\kappa = 3.0$, $\theta = 0.1$, $\xi = 0.4$, $\rho = -0.2$ and initial values $v_0 \sim \mathcal{N}(0.25, 0.02^2)$, $S_0 = 10$. Then, we use this simulation result as the true volatilities of the model based on $S_t$ for the comparison. The stock price and volatility process are shown in Figure 3.1.

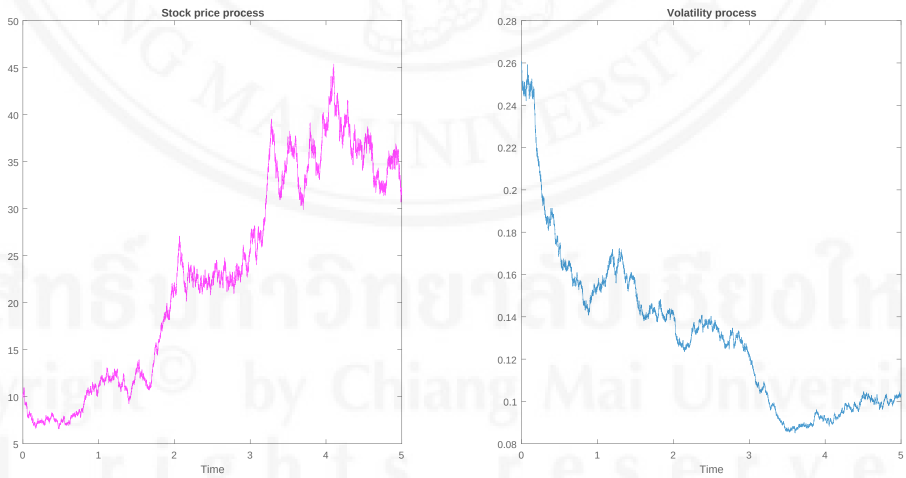


Figure 3.1: Stock price process and volatility process

To apply the SIS particle filter algorithm, we set the number of particles, $N$, as 1000 and the effective sample size threshold $N_T = 2N/3$. We consider four cases of simulations as followed:

Case 1 **Exact parameters case:** The values of parameters are assumed to be known. That is, we set

$$\mu = 0.1,\ \kappa = 3.0,\ \theta = 0.1,\ \xi = 0.4 \text{ and } \rho = -0.2.$$

Case 2 **Good-range set of parameters case:** The values of parameters are assumed to unknown and the true values are contained inside the guessed ranges as follows:

$$\mu \sim U(0.05, 0.15),\ \kappa \sim U(1, 5),\ \theta \sim U(0.05, 0.15),\ \xi \sim U(0.1, 0.7) \text{ and } \rho \sim U(-0.5, 0.1).$$

Case 3 **Bad-range set of parameters case:** The values of parameters are assumed to unknown but the true values are not contained inside the guessed ranges as follows:

$$\mu \sim U(0.6, 0.9),\ \kappa \sim U(10, 15),\ \theta \sim U(0.45, 0.8),\ \xi \sim U(1, 3) \text{ and } \rho \sim U(0.1, 0.8).$$

Case 4 **Wide-range set of parameters case:** The values of parameters are assumed to unknown, the true values are contained inside the guessed ranges, but the ranges are much wider than in Case 2 as follows:

$$\mu \sim U(0.01, 2),\ \kappa \sim U(0.1, 10),\ \theta \sim U(0.01, 0.5),\ \xi \sim U(0.02, 2) \text{ and } \rho \sim U(-0.8, 0.6).$$

Note that $U(a, b)$ denotes the uniform distribution where $a, b$ are lower and upper bounds respectively.

In Case 2, 3 and 4, the hidden processes will be a state vector

$$\mathbf{z}_{t_n} = (\widetilde{v}_{t_n}, \mu_{t_n}, \kappa_{t_n}, \theta_{t_n}, \xi_{t_n}, \rho_{t_n}).$$

Also during the update step, the value of $v_{t_n}$ will be updated from (3.1.7). Note that the rest of parameters is time independent, the values are not update and we encounter the degeneration problem, described in [1]. In this reseach to avoid this deficiency, we assume that the parameter processes are the simple linear process with Gaussian noise, that is

$$\mu_{t_n} = \mu_{t_{n-1}} + \eta^1_{t_n},\ \kappa_{t_n} = \kappa_{t_{n-1}} + \eta^2_{t_n},\ \theta_{t_n} = \theta_{t_{n-1}} + \eta^3_{t_n},\ \xi_{t_n} = \xi_{t_{n-1}} + \eta^4_{t_n} \text{ and } \rho_{t_n} = \rho_{t_{n-1}} + \eta^5_{t_n},$$

where $\eta^1_{t_n}, \eta^2_{t_n}, \eta^3_{t_n}, \eta^4_{t_n}, \eta^5_{t_n} \sim \mathcal{N}(0, 0.01^2)$.

In each case, we plot the simulated values from the SIS particle filter method against the true value, as well as the square error, $|\widetilde{v}_{t_n} - \hat{v}_{t_n}|^2$. The simulation results of Case 1 to 4 are shown in Figure 3.2, Figure 3.3-3.4, Figure 3.5-3.6 and Figure 3.7-3.8 respectively.

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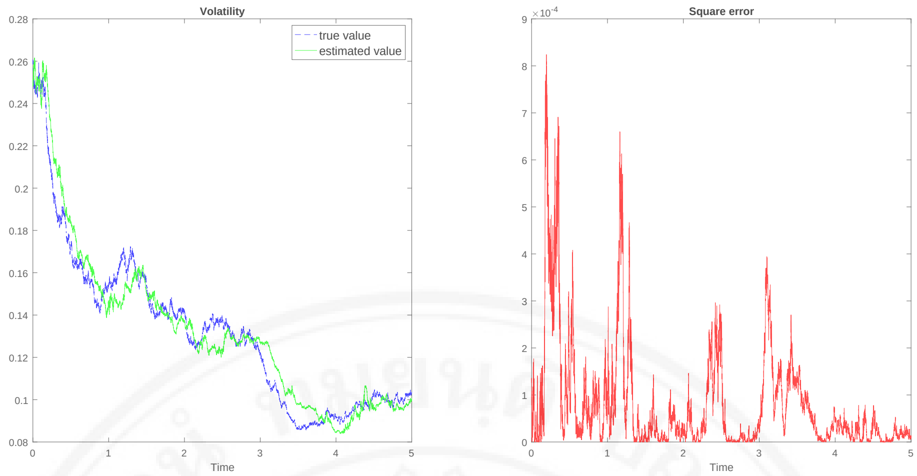


Figure 3.2: Exact parameters case: volatilities and square errors by SIS particle filter

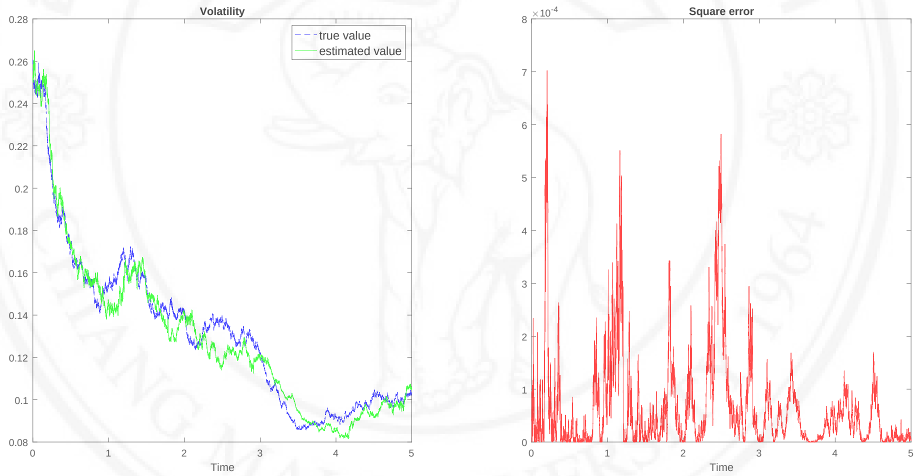


Figure 3.3: Good-range set of parameters case: volatilities and square errors by SIS particle filter

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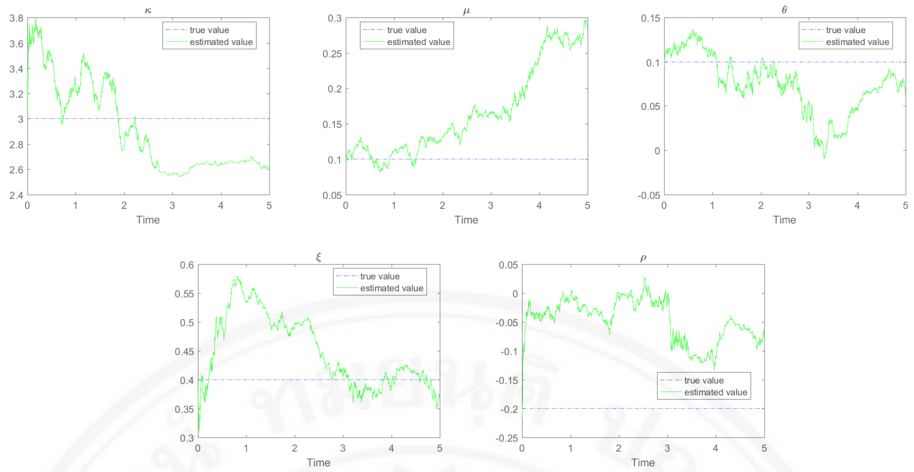


Figure 3.4: Good-range set of parameters case: parameters estimation by SIS particle filter

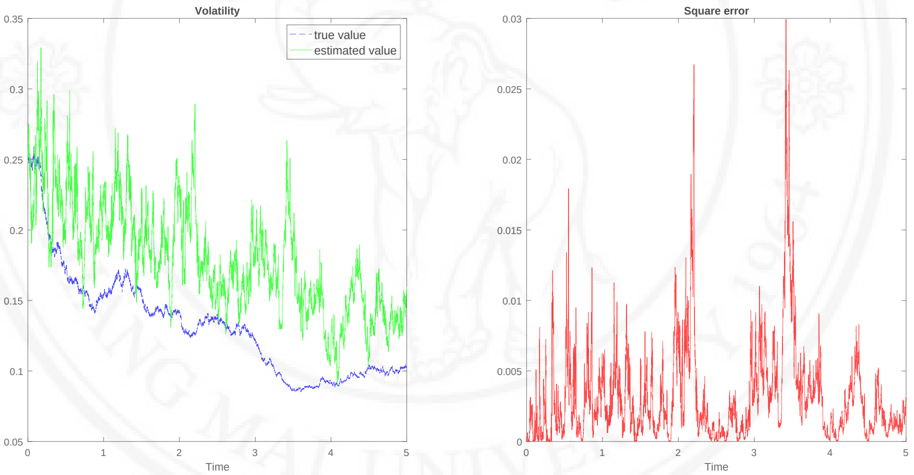


Figure 3.5: Bad-range set of parameters case: volatilities and square errors by SIS particle filter

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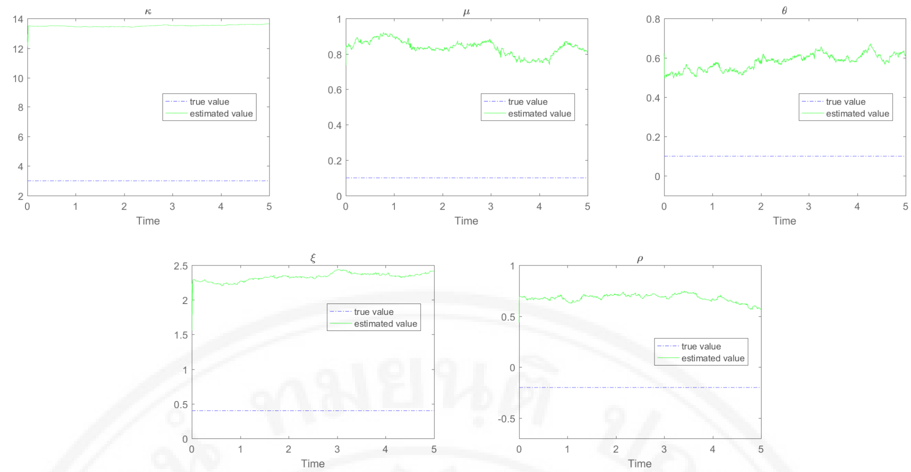


Figure 3.6: Bad-range set of parameters case: parameters estimation by SIS particle filter

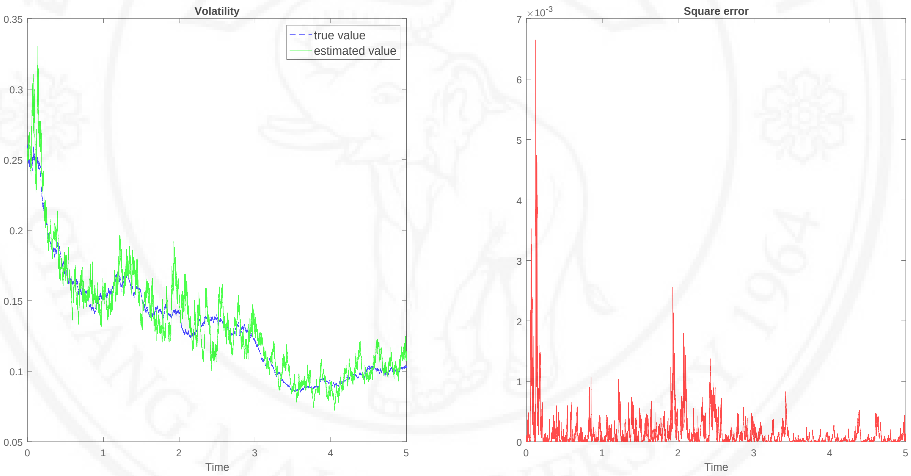


Figure 3.7: Wide-range set of parameters case: volatilities and square errors by SIS particle filter

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**Simulation Studies Analysis**

From the simulation results, we can see that the volatility estimation in Case 1 and 2 are very close to the true volatilities. However, the estimation is not so precise in Case 3 and 4. This suggests that the choice of parameters are also important for the estimation of volatility process. Also, note that the parameter values do not tend to the true values in all cases. Hence, it is suggested to use other estimation technique for parameters estimation, the maximum likelihood estimation (MLE), is proposed in the next section.

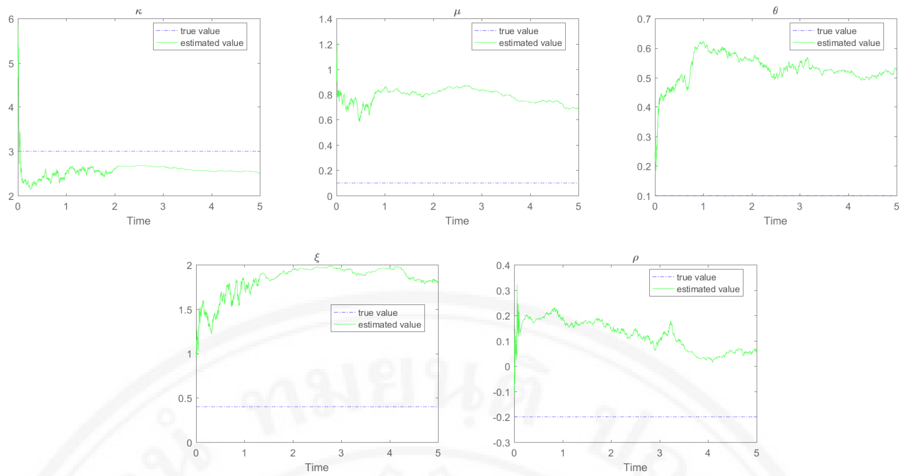


Figure 3.8: Wide-range set of parameters case: parameters estimation by SIS particle filter

## 3.2 Estimation of Model Parameters

In this section, we use the log-price process and estimated volatility process, which obtained from the previous section, to estimate the model parameters in 3/2 model by using the maximum likelihood estimation method. The log-likelihood function for solving this problem is derived.

First, we claim that the volatility process and the log-price process are observed.

**Remark 3.2.1.** *By setting* $\mathbf{x}_{t_n} = \begin{bmatrix} \widetilde{y}_{t_n} & \widetilde{v}_{t_n} \end{bmatrix}^T$*, where* $\widetilde{y}_{t_n}$ *and* $\widetilde{v}_{t_n}$ *are defined in Lemma 3.1.2. We have* $\mathbf{x}_{t_n}|\mathbf{x}_{t_{n-1}}$ *is a normal random vector. To show that* $\mathbf{x}_{t_n}|\mathbf{x}_{t_{n-1}}$ *is a normal random vector, we need to show that* $k_1\widetilde{y}_{t_n} + k_2\widetilde{v}_{t_n}$ *has a normal distribution for* $k_1, k_2 \in \mathbb{R}$*. From (3.1.6)-(3.1.7) and condition on* $\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}$*, we can write* $k_1\widetilde{y}_{t_n} + k_2\widetilde{v}_{t_n}$ *in the form of* $h_1 + h_2\Delta B_{t_n} + h_3\Delta Z_{t_n}$*, where* $h_1, h_2$ *and* $h_3$ *are some constants. Since* $\Delta B_{t_n}$ *and* $\Delta Z_{t_n}$ *are independent normal random variables, the proof is now completed.*

Now, we will derive the probability density function of $\mathbf{x}_{t_n}|\mathbf{x}_{t_{n-1}}$ and the log-likelihood function $L(\boldsymbol{\alpha}|\mathbf{x}_{t_1}, ..., \mathbf{x}_{t_N})$, where $\boldsymbol{\alpha} := (\mu, \kappa, \theta, \xi, \rho)$ is a vector of parameter.

**Proposition 3.2.2.** *Let* $\widetilde{y}_{t_n}$ *and* $\widetilde{v}_{t_n}$ *satisfy the discritized system (3.1.6)-(3.1.7). Then,*

*(a)* $\mathbb{E}(\widetilde{y}_{t_n}|\widetilde{v}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) = \widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}_{t_n})\Delta t_n,$

*(b)* $$\mathbb{E}(\widetilde{v}_{t_n}|\widetilde{y}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) = \widetilde{v}_{t_{n-1}} + \big(\kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}}) - \xi\rho(\mu - \tfrac{1}{2}\widetilde{v}_{t_{n-1}})\widetilde{v}_{t_{n-1}}\big)\Delta t_n + \xi\rho\widetilde{v}_{t_{n-1}}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}),$$

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*(c)* $\operatorname{var}(\widetilde{y}_{t_n}|\widetilde{v}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) = \widetilde{v}_{t_n-1}\Delta t_n$,

*(d)* $\operatorname{var}(\widetilde{v}_{t_n}|\widetilde{y}_{t_n}, \widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) = \xi^2(1-\rho^2)\widetilde{v}^3_{t_{n-1}}\Delta t_n$,

*(e)* $\operatorname{cov}(\widetilde{y}_{t_n}, \widetilde{v}_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}), = \xi\rho\widetilde{v}^2_{t_{n-1}}\Delta t_n$,

*where* $\Delta t_n = t_n - t_{n-1}$.

*Proof.* For (a)-(d), the proofs are obtained in Proposition 3.1.3.
To get (e), we rewrite (3.1.7) as:

$$\widetilde{v}_{t_n} = a_{t_n} + b_{t_n}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}) + c_{t_n}\Delta Z_{t_n},$$

where $a_{t_n} = \widetilde{v}_{t_{n-1}} + \big(\kappa\widetilde{v}_{t_{n-1}}(\theta - \widetilde{v}_{t_{n-1}}) - \xi\rho(\mu - \frac{1}{2}\widetilde{v}_{t_{n-1}})\widetilde{v}_{t_{n-1}}\big)\Delta t_n$, $b_{t_n} = \xi\rho\widetilde{v}_{t_{n-1}}$ and $c_{t_n} = \xi\sqrt{1-\rho^2}\widetilde{v}^{\frac{3}{2}}_{t_{n-1}}$.
Hence, we obtain that

$$\begin{aligned}
\operatorname{cov}(\widetilde{y}_{t_n}, \widetilde{v}_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) &= \operatorname{cov}(\widetilde{y}_{t_n}, a_{t_n} + b_{t_n}(\widetilde{y}_{t_n} - \widetilde{y}_{t_{n-1}}) + c_{t_n}\Delta Z_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \operatorname{cov}(\widetilde{y}_{t_n}, b_{t_n}\widetilde{y}_{t_n} + c_{t_n}\Delta Z_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \operatorname{cov}(\widetilde{y}_{t_n}, b_{t_n}\widetilde{y}_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) + \operatorname{cov}(\widetilde{y}_{t_n}, c_{t_n}\Delta Z_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}).
\end{aligned}$$

Consider

$$\begin{aligned}
\operatorname{cov}(\widetilde{y}_{t_n}, b_{t_n}\widetilde{y}_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) &= b_{t_n}\operatorname{cov}(\widetilde{y}_{t_n}, \widetilde{y}_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= b_{t_n}\operatorname{var}(\widetilde{y}_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \xi\rho\widetilde{v}^2_{t_{n-1}}\Delta t_n, \qquad (3.2.1)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{cov}(\widetilde{y}_{t_n}, c_{t_n}\Delta Z_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) &= \operatorname{cov}(\widetilde{y}_{t_{n-1}} + (\mu - \frac{1}{2}\widetilde{v}_{t_{n-1}})\Delta t_n + \sqrt{\widetilde{v}_{t_{n-1}}}\Delta B_{t_n}, c_{t_n}\Delta Z_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \operatorname{cov}(\sqrt{\widetilde{v}_{t_{n-1}}}\Delta B_{t_n}, c_{t_n}\Delta Z_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) \\
&= \sqrt{\widetilde{v}_{t_{n-1}}}c_{t_n}\operatorname{cov}(\Delta B_{t_n}, \Delta Z_{t_n}) \\
&= 0. \qquad (3.2.2)
\end{aligned}$$

Then, from (3.2.1) and (3.2.2), we have

$$\operatorname{cov}(\widetilde{y}_{t_n}, \widetilde{v}_{t_n}|\widetilde{y}_{t_{n-1}}, \widetilde{v}_{t_{n-1}}) = \xi\rho\widetilde{v}^2_{t_{n-1}}\Delta t_n.$$

□

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**Remark 3.2.3.** *The normal random vector* $\mathbf{x}_{t_n}|\mathbf{x}_{t_{n-1}}$ *has the probability density function*

$$p(\mathbf{x}_{t_n}|\mathbf{x}_{t_{n-1}}) = \frac{1}{\sqrt{4\pi^2|\mathbf{\Sigma}_{t_n}|}} \exp\big(-\frac{1}{2}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})^T\mathbf{\Sigma}_{t_n}^{-1}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})\big),$$

*where* $\boldsymbol{\mu}_{t_n}$, $\mathbf{\Sigma}_{t_n}$ *are the mean vector and covariance matrix of* $\mathbf{x}_{t_n}$, *which all elements are defined in Proposition 3.2.2.*

**Theorem 3.2.4.** *Let* $(\mathbf{x}_{t_1}, ..., \mathbf{x}_{t_N})$ *be a vector of* $N$*-observed data and* $\boldsymbol{\alpha}$ *be a vector of parameter. The log-likelihood function of 3/2 stochastic volatility model is*

$$L(\boldsymbol{\alpha}|\mathbf{x}_{t_1}, ..., \mathbf{x}_{t_N}) = -\sum_{n=1}^{N}\Bigg(\ln(\sqrt{4\pi^2|\mathbf{\Sigma}_{t_n}|}) + \frac{1}{2}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})^T\mathbf{\Sigma}_{t_n}^{-1}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})\Bigg), \quad (3.2.3)$$

*where* $\mathbf{x}_{t_n}$, $\boldsymbol{\mu}_{t_n}$ *and* $\mathbf{\Sigma}_{t_n}$ *are defined in Remark 3.2.3.*

*Proof.* By Remark 3.2.3 and by conditional probability and Markov property, we have

$$\begin{aligned}
L(\boldsymbol{\alpha}|\mathbf{x}_{t_1}, ..., \mathbf{x}_{t_N}) &= \ln p(\mathbf{x}_{t_1}, ..., \mathbf{x}_{t_N}|\boldsymbol{\alpha}) \\
&= \ln\prod_{n=1}^{N} p(\mathbf{x}_{t_n}|\mathbf{x}_{t_1}, ..., \mathbf{x}_{t_{n-1}}, \boldsymbol{\alpha}) \\
&= \ln\prod_{n=1}^{N} p(\mathbf{x}_{t_n}|\mathbf{x}_{t_{n-1}}, \boldsymbol{\alpha}) \\
&= \sum_{n=1}^{N}\ln p(\mathbf{x}_{t_n}|\mathbf{x}_{t_{n-1}}, \boldsymbol{\alpha}) \\
&= \sum_{n=1}^{N}\Bigg(\ln\big(\frac{1}{\sqrt{4\pi^2|\mathbf{\Sigma}_{t_n}|}}\exp\big(-\frac{1}{2}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})^T\mathbf{\Sigma}_{t_n}^{-1}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})\big)\big)\Bigg) \\
&= \sum_{n=1}^{N}\Bigg(\ln(\frac{1}{\sqrt{4\pi^2|\mathbf{\Sigma}_{t_n}|}}) - \frac{1}{2}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})^T\mathbf{\Sigma}_{t_n}^{-1}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})\Bigg) \\
&= -\sum_{n=1}^{N}\Bigg(\ln(\sqrt{4\pi^2|\mathbf{\Sigma}_{t_n}|}) + \frac{1}{2}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})^T\mathbf{\Sigma}_{t_n}^{-1}(\mathbf{x}_{t_n}-\boldsymbol{\mu}_{t_n})\Bigg).
\end{aligned}$$

Therefore, this proof is complete. □

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**Simulation Studies of Parameter Estimation**

In this simulation study, we select the observed data, volatility process and log-price process, from Case 2 in previous section. The data is shown in Figure 3.3.

Now, we input these data in (3.2.3) to get the log-likelihood function for our model as in Theorem 3.2.4. Finally, we find the maximum likelihood estimators by using the command "fminsearch" in MATLAB program..

We consider two simulation studies as follows:

**Simulation study 1:** We assume that $\mu$ and $\theta$ are known, $\mu = \theta = 0.1$, and we need to estimate the rest of parameters. In order to compare the estimation results, we estimate the values of parameters based on the number of observed data as follows:

Case 1: 5,000 observed data ($\Delta t_n = 0.001$),

Case 2: 10,000 observed data ($\Delta t_n = 0.0005$),

Case 3: 20,000 observed data ($\Delta t_n = 0.00025$).

Table 3.1: Estimated values of $\kappa$, $\xi$ and $\rho$ with estimated data

| Parameter | True value | 5,000 obs. data | | 10,000 obs. data | | 20,000 obs. data | |
|---|---|---|---|---|---|---|---|
| | | Est. val. | Abs. err. | Est. val. | Abs. err. | Est. val. | Abs. err. |
| $\kappa$ | 3.0 | 5.4024 | 2.4024 | 5.4003 | 2.4003 | 5.3218 | 2.3218 |
| $\xi$ | 0.4 | 1.7710 | 0.7710 | 0.8802 | 0.4802 | 0.6895 | 0.2895 |
| $\rho$ | -0.2 | -0.6572 | 0.4572 | -0.6149 | 0.4149 | -0.5494 | 0.3494 |

**Simulation study 2:** All parameters are assumed to unknown. In order to compare the estimation results, we estimate the values of parameters based on the number of observed data as follows:

Case 1: 5,000 observed data ($\Delta t_n = 0.001$),

Case 2: 10,000 observed data ($\Delta t_n = 0.0005$),

Case 3: 20,000 observed data ($\Delta t_n = 0.00025$).

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In each case, we compute the estimated values from maximum likelihood estimation method against the true values, as well as the absolute errors, $|\boldsymbol{\alpha}_{t_n} - \hat{\boldsymbol{\alpha}}_{t_n}|$. The results of Simulation study 1 and Simulation study 2 are shown in Table 3.1 and 3.2 respectively. Both simulation studies show that the estimation obtained from the maximum likelihood method, based on the log-likelihood function in (3.2.3), works fairly well.

Table 3.2: Estimated values of $\mu$,$\kappa$, $\theta$, $\xi$ and $\rho$ with estimated data

| Parameter | True value | 5,000 obs. data | | 10,000 obs. data | | 20,000 obs. data | |
|---|---|---|---|---|---|---|---|
| | | Est. val. | Abs. err. | Est. val. | Abs. err. | Est. val. | Abs. err. |
| $\mu$ | 0.1 | 0.2956 | 0.1956 | 0.1083 | 0.0083 | 0.2901 | 0.1901 |
| $\kappa$ | 3.0 | 5.4121 | 2.4121 | 4.2959 | 1.2959 | 5.4272 | 2.4272 |
| $\theta$ | 0.1 | 0.1002 | 0.0002 | 0.0953 | 0.0047 | 0.1001 | 0.0001 |
| $\xi$ | 0.4 | 0.5638 | 0.1638 | 0.7660 | 0.3660 | 0.4477 | 0.0477 |
| $\rho$ | -0.2 | -0.4541 | 0.2541 | -0.5827 | 0.3827 | -0.1694 | 0.0306 |

In Simulation study 1, the errors decrease as the number of observed data increase. However, it seems that the absolute errors do not depend on the number of observed data in Simulation study 2. This may due to the number of parameters that are estimated in Simulation study 1 is less than that in Simulation study 2.

For the comparison to real data, we use the maximum likelihood estimation method with the data as in Figure 3.1. The results are shown in the following tables.

Table 3.3: Estimated values of $\kappa$, $\xi$ and $\rho$ with real data

| Parameter | True value | 5,000 obs. data | | 10,000 obs. data | | 20,000 obs. data | |
|---|---|---|---|---|---|---|---|
| | | Est. val. | Abs. err. | Est. val. | Abs. err. | Est. val. | Abs. err. |
| $\kappa$ | 3.0 | 2.8562 | 0.1438 | 3.1045 | 0.1045 | 3.0566 | 0.0566 |
| $\xi$ | 0.4 | 0.4679 | 0.0679 | 0.4612 | 0.0612 | 0.4222 | 0.0222 |
| $\rho$ | -0.2 | -0.3063 | 0.1063 | -0.2202 | 0.0202 | -0.2089 | 0.0089 |

Table 3.4: Estimated values of $\mu$,$\kappa$, $\theta$, $\xi$ and $\rho$ with real data

| Parameter | True value | 5,000 obs. data | | 10,000 obs. data | | 20,000 obs. data | |
|---|---|---|---|---|---|---|---|
| | | Est. val. | Abs. err. | Est. val. | Abs. err. | Est. val. | Abs. err. |
| $\mu$ | 0.1 | 0.1692 | 0.0692 | 0.0816 | 0.0184 | 0.0932 | 0.0068 |
| $\kappa$ | 3.0 | 3.2857 | 0.2857 | 3.0709 | 0.0709 | 2.9179 | 0.0821 |
| $\theta$ | 0.1 | 0.0909 | 0.0091 | 0.1190 | 0.0190 | 0.1012 | 0.0012 |
| $\xi$ | 0.4 | 0.4762 | 0.0762 | 0.4422 | 0.0422 | 0.4267 | 0.0267 |
| $\rho$ | -0.2 | -0.3733 | 0.1733 | -0.1767 | 0.0233 | -0.2716 | 0.0716 |

We can see that the estimated value of model parameters with true volatility and estimated volatility from SIS particle filter method have nearly the same value, it can be guarantee that the particle filter method is a quite good method for volatility estimation in our model.

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