CHAPTER 2

Theory

> "We have answered Eddington's question, What appliance can pierce through the outer layers of a star and test the condition within? The answer is **Asteroseismology**, the songs of the stars, the real music of the spheres."
>
> Aerts et al. (2010)

2.1 Stellar Structure and Evolution

2.1.1 Stellar Structure Equations

A star is held together by gravity, which compresses everything towards the centre. On the contrary, thermal and radiation pressures sustain an ordinary star and prevent its total collapse. They expand the star layers outward. In any given shell of a star on the main-sequence, there is a balance between thermal or radiation pressure pushing outward and gravity pressing downward. This balance in the star is called *hydrostatic equilibrium*.

Pressure gradient

To calculate how pressure must change as a function of depth, consider an element in cylindrical shape located from the centre of the star at the distance r as shown in Figure 2.1. $dV = dAdr$ is its volume, dr is its height, dA is its cross section area, $dm = \rho dAdr$ is mass of the element, and $\rho = \rho(r)$ is density of the gas as function of radius r.

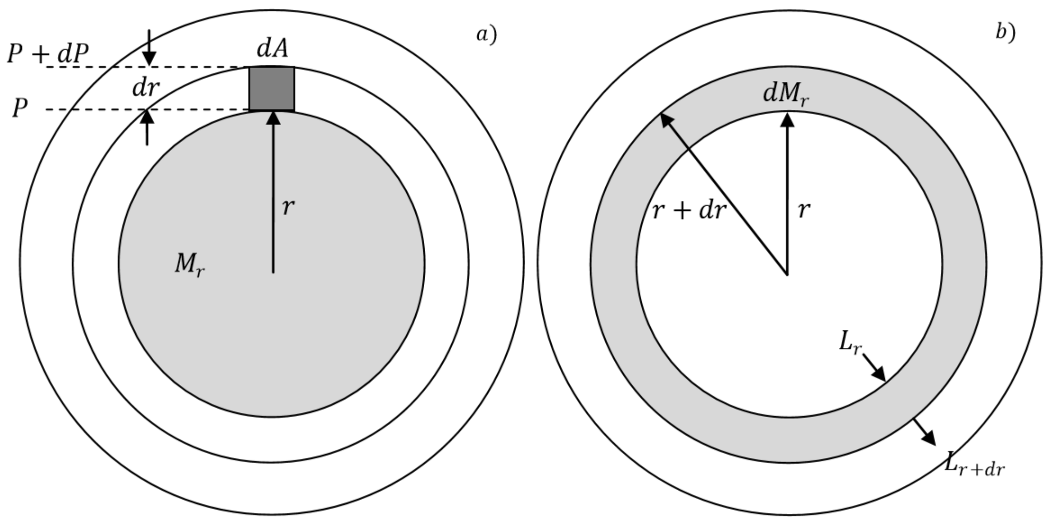

Figure 2.1: Diagrams of a pressure gradient inside the stars. Consider a cylindrical element from the centre of the star at the distance r (Panel a). The mass of a spherical shell M_r comes from its volume and density (Panel b). The energy flowing out of a shell is the difference between the energy produced within it and the energy flowing into it.

The gravitational force acting on the cylindrical element is

$$dF_g = -\frac{GM_r dm}{r^2} = -\frac{GM_r \rho}{r^2} dA dr, \tag{2.1}$$

where M_r is the mass inside radius r. The total pressure force acting on the volume element is

$$dF_p = PdA - (P + dP)dA = -dPdA. \tag{2.2}$$

For the equilibrium condition, the net force acting on the cylindrical element must be zero;

$$0 = dF_g + dF_p. \tag{2.3}$$

Substitute Equations 2.1 and 2.2 into the equilibrium condition, we have:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} = -\rho g, \tag{2.4}$$

where $g = GM_r/r^2$ is the local acceleration of gravity as a function of r. G is gravitational constant and M_r is cumulative mass inside the shell of radius r. In order for a star to be in equilibrium, the gravity force must be resisted by a pressure gradient dP/dr. Moreover, Equation 2.4 indicates that pressure decreases with increasing radius; the pressure in the interior is necessary larger than that near the surface.

Mass conservation

For symmetric star, consider a spherical shell located at the distance r from the centre of the star, its mass is dM_r and its thickness is dr. If ρ is the local gas density and $dV = 4\pi r^2 dr$ is the volume of the shell, the mass of the shell is given by

$$dM_r = \rho(4\pi r^2 dr). \tag{2.5}$$

Therefore, the cumulative mass changes as a function of radius according to the mass conservation equation:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho. \tag{2.6}$$

Luminosity gradient

All of the energy generated by stellar materials must be considered in order to determine the luminosity of a star. The total luminosity changes due to mass dm is given by

$$dL = \epsilon dm, \tag{2.7}$$

where ϵ is the total energy from nuclear reactions and gravity ($\epsilon = \epsilon_{nuclear} + \epsilon_{gravity}$). We consider a thickness dr of star's thin shell and $dm = dM_r = \rho dV = 4\pi r^2 \rho dr$, we obtain the luminosity L as a function of distance r given by

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon, \tag{2.8}$$

where luminosity L_r is the interior luminosity due to total amount of energy passing through each spherical layer from the centre of the star at the distance r. L_r includes convective, radiative and conductive energy transports.

Radiative temperature gradient

First consider the radiation transport, where the radiation pressure gradient is given by

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c} F_{rad}, \tag{2.9}$$

where F_{rad} is the radiative flux, ρ is density, c is speed of light and $\bar{\kappa}$ is opacity. From the relationship between radiation pressure and temperature $P_{rad} = aT^4/3$, the radiation pressure gradient may be expressed as

$$\frac{dP_{rad}}{dr} = \frac{4}{3} a T^3 \frac{dT}{dr} \tag{2.10}$$

$$-\frac{\bar{\kappa}\rho}{c} F_{rad} = \frac{4}{3} a T^3 \frac{dT}{dr}, \tag{2.11}$$

where a is the radiation constant. $F_{rad} = L_r/4\pi r^2$ is the local radiative flux of the star at radius r. Thus, the temperature stratification for radiative transport is presented by

$$\frac{dT}{dr} = -\frac{3}{4acT^3}\frac{\kappa\rho}{c}\left(\frac{L_r}{4\pi r^2}\right). \tag{2.12}$$

When the opacity or the flux increases, the temperature gradient must decrease.

2.1.2 Hertzsprung-Russell (H-R) Diagram

The Hertzsprung-Russell (H-R) diagram is a graph which presents the brightness and temperature characteristics of stars. It provides important hints to the evolution of stars. The diagram was first plotted by Henry Norris Russell in 1913, but it has been done independently at the same time by Ejnar Hertzsprung (Rees, 2001). Russell published the diagram which records the observed properties of more than 200 stars, most of which located in a diagonal region reaching from the upper left-hand corner, i.e., the position of the hot and bright O type stars, to the lower right-hand corner, i.e., where the cool and dim M type stars reside. This concentrated region is the *main sequence*. The giant stars occupy the region above the lower main sequence, with the supergiant, in the extreme upper right-hand corner. The White Dwarfs lie well below the main sequence in the left corner (Carroll and Ostlie, 2007). The HR-diagram is a fundamental in astronomy in many ways:

1) It demonstrates the fundamental characteristics of stars: 90% of a star's lifetime is on the main sequence phase. Its luminosity, radius, and surface temperature are directly determined by its mass. Their correlation is evidenced by the HR-diagram.

2) It is a way to classify and compare populations of stars. For example, the population of nearby stars to the Sun are located on the part of the main sequence below the Sun's position. In the opposite, population of apparently bright stars in the sky all are located above the Sun's position.

3) It can be used as a distance-measuring tool: if a star is located on the main sequence, from it apparent brightness and the inverse-square law one may estimate its distance.

4) It can be used as a tracer of stellar evolution: theoretical evolutionary tracks on the H-R diagram can predict the changes in luminosity and surface temperature of the stars. They also can be compared with the properties of observed stars. This technique is very important to estimate the ages of star clusters.

2.1.3 Stellar Evolution

The Universe began with the Big Bang 13.7 billion years ago. The first stars formed with almost no metal content; $Z = 0$. The next generation of stars that formed was extremely metal-poor, having very low but non-zero values of Z. Each generation of newborn stars resulted in higher proportions of heavier elements, leading to metal-rich stars. The population of stars can be roughly divided into three generations: Population III are the original stars that formed immediately after the Big Bang, Population II are metal-poor stars with $Z > 0$, and Population I are metal-rich stars (Carroll and Ostlie, 2007). Stellar evolution is a theory which describes the change in the structure and compositions of an individual star as it ages. Moreover, the theory of evolution needs to be able to explain the position of stars within the H-R diagram.

Pre-main-sequence stars

Stars form from loose gas and dust between the stars called *interstellar material*. Once a dense molecular cloud collapses under the effect of gravity, i.e. its own weight, it is characterized by the *free-fall timescale* (t_{ff}) given by

$$t_{ff} = \left(\frac{3\pi}{32}\frac{1}{G\rho_0}\right)^{1/2}, \tag{2.13}$$

where this timescale depends on the original density ρ_0 of the spherical molecular cloud. As long as the original density was uniform, the cloud will take the same time to collapse. The protostar is formed by the rate at which the star can thermally adjust to collapse called the *Kelvin-Helmholtz timescale* (t_{KH}):

$$t_{KH} = \frac{\Delta E_g}{L_\odot}, \tag{2.14}$$

where E_g is the energy radiated away during collapse and $L_\odot$ is luminosity of the Sun. Since $t_{KH} \gg t_{ff}$, protostar proceeds at a slower rate than the free-fall collapse. A solar mass star requires about 40 Myr to contract to its main-sequence stage. The cloud will collapse when its mass is more than a certain criterion called the *Jeans criterion*:

$$M > M_J \propto \frac{T^{3/2}}{\rho^{1/2}\mu^{3/2}}, \tag{2.15}$$

where μ is mean molecular weight, ρ is mean density, and T is the temperature of the cloud. This free-fall collapse will continue in isothermal process. An adiabatic contraction happens as long as the free-fall time equal to the thermal relaxation time. When the process arrived to a natural end, it becomes *protostars* in stellar masses. Their internal temperature are rather low and the opacity is high.

The protostar arrives at hydrostatic equilibrium and enters its *pre-main-sequence* phase after the dynamical contraction. The further contraction of the star indicates that the star decreases in luminosity but maintains the same effective temperature. In this process, the temperature in the star slightly increases and the opacity decreases, resulting the convective zone begins to move away from the centre of the star. This step is called *radiative contraction*.

The increase of temperature at the core produces the *proton-proton reaction* or PP reaction. This reaction transforms H to ^{2}H, and immediately burnt into ^{3}He. Since ^{3}He is still not enough for hydrogen burning, the full PP chain cannot be completed. As a result, the temperature sensitivity of the nuclear reactions is high and the core will becomes a *convective core*. In stars with the masses below than $1.1M_\odot$, this convective core will vanish when the PP chain has all its intermediate elements at equilibrium. For more massive stars, they quickly change to hydrogen burning via the CNO cycle. This cycle is more temperature sensitive compare with that in the PP chain at equilibrium.

During the pre-main-sequence phase, the accretion continues on a thermal time scale or Kelvin-Helmholtz time scale. Pre-main-sequence stars with masses in the range of 0.8 and $1.6M_\odot$ end their accretion phase and light up in the H-R diagram called a *T Tauri star*. Pre-main-sequence stars with masses in the range of 1.6 and $9M_\odot$ end their accretion phase before they arrive the main sequence, eventually they become *Herbig Ae/Be stars*. protostars with masses larger than $9M_\odot$ rapidly evolve unto the main sequence.

Zero-age-main sequence stars

The star with masses larger than $0.08M_\odot$, hydrogen in the core is burning in hydrostatic equilibrium and produce the energy. The contraction will stop, the star arrives at thermal equilibrium state and it lies on the *zero-age main sequence* (ZAMS). Protostars with masses less than $0.08M_\odot$ never arrive the ZAMS because they cannot produce a high temperature core to burn hydrogen in equilibrium. They will simply cool slowly with time, eventually become *brown dwarfs*.

Main sequence and post-main-sequence stars

The stars spend more than half of their life on the main sequence. In this phase, they burn H into He on a nuclear timescale as given by

$$t_{nuclear} = \frac{\epsilon q M c^2}{L} \tag{2.16}$$

$$t_{nuclear} \sim 10^{10}\frac{M}{L} \text{ years,} \tag{2.17}$$

where ϵ is a part of stellar mass which is converted into energy in the process of nuclear reactions. It is around 0.7% for hydrogen fusion. q is the small fraction of the stellar mass that can take part in the nuclear burning. This time scale represents the period of time that the star can use nuclear fusion to produce energy.

The interior structure of the stars in terms of convective, radiative, and diffusive energy transports is very different depending on stellar mass. The initial chemical composition is the next important factors in the details of the evolution. The star has arrived the *terminal-age main sequence* (TAMS) when the central hydrogen is exhausted. Then, the stars produce energy by burning hydrogen shell; its helium core starts shrinking, while the outer layers of the star expand, causing the star become a large, cool, luminous *red giant star*.

Low mass stars

The objects having low-mass central helium burning start to be the stars with masses $2.3M_\odot \leq M \leq 9M_\odot$. They gradually start helium burning in the core at the end of the terminal-age main sequence (TAMS). The stars radiate energy by helium and hydrogen-shell burning after the central helium is exhausted and become *Asymptotic Giant Branch* (AGB). In this stage, the re-ignition and extinction of the helium shell burning cause nuclear burning involves thermal pulses. This lead to complex nuclear reactions. The stars with initial masses $6M_\odot \leq M \leq 9M_\odot$ may reach phases of carbon burning and end with an O, Ne, Mg *white dwarf*.

Stars with initial masses about $6M_\odot$ for which this occurs is uncertain and depends on rotational mixing and the mass loss. A dust-driven wind and large-amplitude pulsations in the stars on the AGB are the main reasons of release amount of their mass. The envelope expansion with the shell burning causes the outer layers are loosely bound. The hydrogen-burning shell is extinguished and the mass loss stops, the star change its life to *post-Asymptotic Giant Branch* (post-AGB) phase. The remaining outer layers called *planetary nebula*, which are rapidly expelled and the resulting ejected shell radiates. The remaining inner part is the hot core, which rapidly cools and become a *white dwarf* star.

Stars with masses lower than $0.5M_\odot$ have not yet evolved off the main sequence. The core helium burning cannot be ignited because the temperatures at their cores are not high enough, so the end of their life will be He white dwarfs. Stars with an initial masses $0.5M_\odot \leq M \leq 2.3M_\odot$ have a degenerate helium core after the main sequence. After several Gyr later, they will reach the TAMS depending on their initial masses. After the

TAMS, the core continuously contracts until reaches the helium burning phase, then a helium flash starts. The star is burning hydrogen in a shell and helium in its core. Eventually, they located near the red-giant branch.

High mass stars

On the main sequence, stars with masses larger than $25M_{\odot}$ have very strong radiatively driven winds which cause the stars lose their mass. Outburst and instabilities occur because the radiation pressure is very strong and unstable. The stars are called *luminous blue variables* and, after they released a large amount of their hydrogen shell, they become *Wolf-Rayet* stars. They finish their life as supernovae explosions, and may be origin of *stellar black holes*.

2.1.4 Star Clusters

A stellar cluster is a group of stars which formed almost synchronously, under the same physical conditions, located at the same distance. The study of such star clusters has played a crucial part in the development of our understanding of stellar evolution. Star clusters are very gigantic in the compactness of their structure and their luminosity. Star clusters are classified in two types: *globular clusters* are more compact, luminous and metal-poor objects. They contain very old Population II stars which a few hundred million years younger than the universe. While, *open clusters* are less compact, less luminous and more metal-rich clusters. They commonly contain hot young blue Population I stars with the ages of tens of millions years.

Open Clusters

An open cluster is a group of stars which are formed from the same giant molecular cloud and have approximately the same age. Some open clusters may contain a few thousand stars. Nowadays, astronomers have discovered more than 1,100 open clusters within the Milky Way Galaxy, and many more are speculated to exist. Stars in open clusters loosely hold together by mutual gravitational attraction. When they orbit around the galactic center, they are disrupted by the stars in other clusters and gas clouds in galactic plane, resulting in rejection of cluster members (Karttunen et al., 2007). Open clusters are commonly found in the disk of the Galaxy and hence they are also sometimes referred to as *Galactic clusters*.

The properties of an open cluster can be usefully described by Trumpler classification (Trumpler, 1930):

1) Its degree of concentration as Roman numeral, where lower numbers indicate higher degrees central of concentration

2) The range in brightness of its stars measured on a scale 1…3, where lower numbers indicate a smaller range of stellar brightness;

3) Its richness, specified as p, m or r depending on whether the system is poor (containing less than 50 stars), moderate (50-100 stars), or rich (more than 100 stars).

Open clusters play an important role to understand galactic astronomy. Since they contain bright, blue stars, open clusters are testing grounds for investigating the properties of high mass stars. The special distribution of open clusters yield a useful probe of the structure of the Galactic disk: the youngest systems analyze the spiral structure of the Galaxy, while the older systems trace the kinematics of the outer Galaxy. Since it is believed that many of stars in the disk of Galaxy originated in the open clusters, the properties of these systems must dictate many of the properties of the stellar disk as a whole (Binney and Merrifield, 1998).

2.1.5 Isochrones and Cluster Ages

Theoretically, the evolutionary tracks of stars in different masses having the same composition can be computed to understand their evolution. The position of each star can be plotted on the H-R diagram. The curve presents a population of stars of the same age called an *isochrone* as shown in Figure 2.2. The relative number of stars at each location on the isochrone depends on the number of stars in each mass range within the cluster, combined with the different rates of evolution during each phase. Therefore, star counts in a colour-magnitude diagram can shine light on the timescales involved in stellar evolution.

As the cluster ages, when molecular cloud collapse, the most massive and less abundant stars will arrive on the main sequence first. Before the lowest-mass stars even reach the main sequence, the most massive ones have already evolved into the red giant region, perhaps even undergoing supernova explosions. Since core hydrogen burning lifetimes are inversely related to mass, continued evolution of the cluster means that the main sequence turn-off point. The point is the position on the H-R diagram where stars in the cluster are leaving off the main sequence. In this stage, the star cluster becomes redder and less luminous with time. Consequently, it is possible to estimate the age of cluster by the location of the uppermost point of its main sequence. This fundamental technique is an important tool for determining ages of stars, clusters, our Milky Way Galaxy, and other galaxies with observable clusters, and even for establishing a lower boundary on the age of the universe (Carroll and Ostlie, 2007).

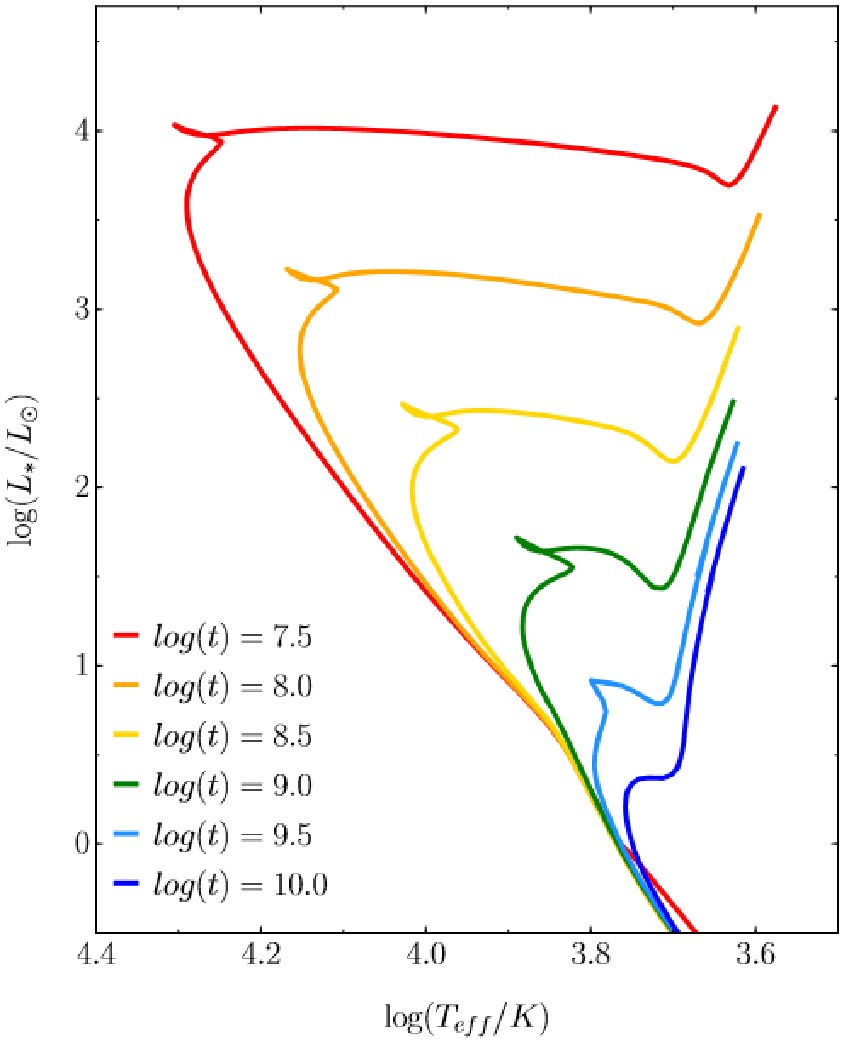

Figure 2.2: Isochrones for a cluster of stars with masses between $0.8M_\odot$ and $8.6M_\odot$ plotted by using PARSEC: the database of stellar tracks and isochrones computed with the PAdova and TRieste Stellar Evolution Code (Bressan et al., 2012).

2.2 Introduction of Astroseismology

By studying our closest star, the Sun, helioseismology provides a great tool in probing the precise interior structural model of the Sun. Asteroseimology aims to study the interior structures of different types of stars by analysing their oscillations. Stellar oscillations provide an exclusive way to probe the internal processes and properties in the stars, because their interior properties cause the frequencies, their modes, amplitudes, and phases from observations.

2.2.1 Stellar 3-D Oscillations in General

Theoretically, stars are three-dimensional spheres, so they can be described by spherical harmonics $Y_l^m(\theta, \phi)$ in spherical coordinates given by

$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) \exp(im\phi), \tag{2.18}$$

where $P_l^m(\cos\theta)$ are Legendre polynomials given by

$$P_l^m(\cos\theta) = \frac{1}{2^l l!}(1-\cos^2\theta)^{\frac{m}{2}} \frac{d^{l+m}}{d\cos^{l+m}\theta}(\cos^2\theta - 1)^l. \tag{2.19}$$

Note that the spherical harmonics are usually defined such that the integral of $| Y_l^m |^2$ over the unit sphere equals 1, where c_{lm} is the normalization constant

$$c_{lm} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}. \tag{2.20}$$

Stellar oscillations are described by considering the reaction of the star to a small perturbation of its equilibrium state. Usually, the pulsation modes have nodes in three orthogonal directions, represented by the distance r to the centre, co-latitude θ and longitude ϕ. In the case of a spherically symmetric star in equilibrium, the solutions to the equations of motion for the displacement vector in the (r, θ, ϕ) directions are given by

$$\xi_r(r,\theta,\phi,t) = a(r)Y_l^m(\theta,\phi)\exp(-2i\pi\nu t) \tag{2.21}$$

$$\xi_\theta(r,\theta,\phi,t) = b(r)\frac{Y_l^m(\theta,\phi)}{\theta}\exp(-2i\pi\nu t) \tag{2.22}$$

$$\xi_\phi(r,\theta,\phi,t) = \frac{b(r)}{sin\theta}\frac{\partial Y_l^m(\theta,\phi)}{\partial\phi}\exp(-2i\pi\nu t), \tag{2.23}$$

where the co-latitude θ is measured from the symmetric pole, while the latitude is measured from the equator. The nodes are concentric shells at constant r, cones of constant θ and planes of constant ϕ. ξ_r, ξ_θ and ξ_ϕ are the displacements in three-dimensions, $a(r)$ and $b(r)$ are the amplitudes, ν is the oscillation frequency. In this expression above, it is assumed that the star has slow rotation and magnetism or tidal forces due to a close companion are ignored.

In most pulsating stars, the symmetric axis coincides with the rotation axis. As the stars are assumed to be three-dimensional spheres, there are three quantum numbers to specify these modes: n represents the number in the radial direction and is called the *order* of the mode; the *degree l* is the total number of nodal lines of the displacement field at the stellar surface; m is the *azimuthal order* of the mode, where $|m|$ specifies how many of the surface nodal lines are meridian lines. So the number of surface nodal lines that are lines of co-latitude is equal to $l - |m|$. The values of m range from l to $+l$, so there are $2l+1$ modes for each degree l. An extensive and detailed description of asteroseismology and stellar pulsations is available in the books by Unno et al. (1979), Cox (1980) and Aerts et al. (2010).

2.2.2 Pulsation Modes: Radial & Non-Radial

In theory of stellar oscillations, pulsation modes can be divided into two different types: the radial and non-radial pulsation modes. The simplest form of stellar pulsations is *radial modes* with degree $l = 0$, $m = 0$. There is an infinite number of radial modes in a star, the simplest one is the *fundamental radial mode*. In this basic mode, all parts of the star periodically expand together and contract together, while the stellar envelope keeps its spherical shape. The next-simplest mode is the *first overtone*. In this mode there is *nordal sphere* in the star, called *node*, which is a layer that does not move, when the part of the star inside this sphere is contracting the part outside are expanding and vice versa. In the *second overtone* mode, there are two nodes inside the star.

Radial modes are the usual mode of the large-amplitude pulsating stars, for example, Cepheid variables, Mira stars and RR Lyrae stars. In the cases of the RR Lyrae and Cepheid variable stars, there are also RRd stars and double-mode Cepheids, respectively. The ratios of the first overtone period to the fundamental period in various types of stars are different; it is 0.71 for Cepheids and 0.77 for δ Scuti stars. In the case of organ pipes and strings the ratios are 0.33 and 0.5, respectively (Aerts et al., 2010). The difference is very useful for asteroseismology. The ratio would be the same to that in the musical instruments if the star has uniform temperature and chemical composition, therefore the sound speed is constant.

The larger ratios in the Cepheids and δ Scuti stars are results of the sound speed gradient in these stars. The differences between the Cepheid and δ Scuti ratios are a result of Cepheid giant star being more centrally condensed than the hydrogen core-burning δ Scuti star. Therefore, by observing simultaneously two pulsation frequencies, e.g. first harmonic and fundamental mode, we obtain a first look into the stellar interiors. Radial mode pulsation produces significant changes in temperature, luminosity and radial velocity of the stars; these can be observed and provide information about radial pulsation.

The more complex forms of stellar oscillations are the *non-radial* modes, the star changes its shape but not volume. There is a infinite set of modes, corresponding to the three different coordinate axes on the star. The simplest of the non-radial modes is the axisymmetric dipole mode with $l = 1$, $m = 0$. For this mode the equator is a node; the northern hemisphere expands while the southern hemisphere contracts and vice versa. This mode can be described with the simple cosine dependence of $P_1^0(\cos\theta) = \cos\theta$. The star seems to oscillate in space from the observer's point of view because there is no change to the circular cross-section of the star. Figure 2.3 represents possible cases of octupole modes with $l = 3$ showing the modes from different viewing angles. Table 2.1 presents several classes of pulsating stars and some of their properties including variation period, population and spectral types. We indicate where radial (R) and non-radial (NR) oscillations occur.

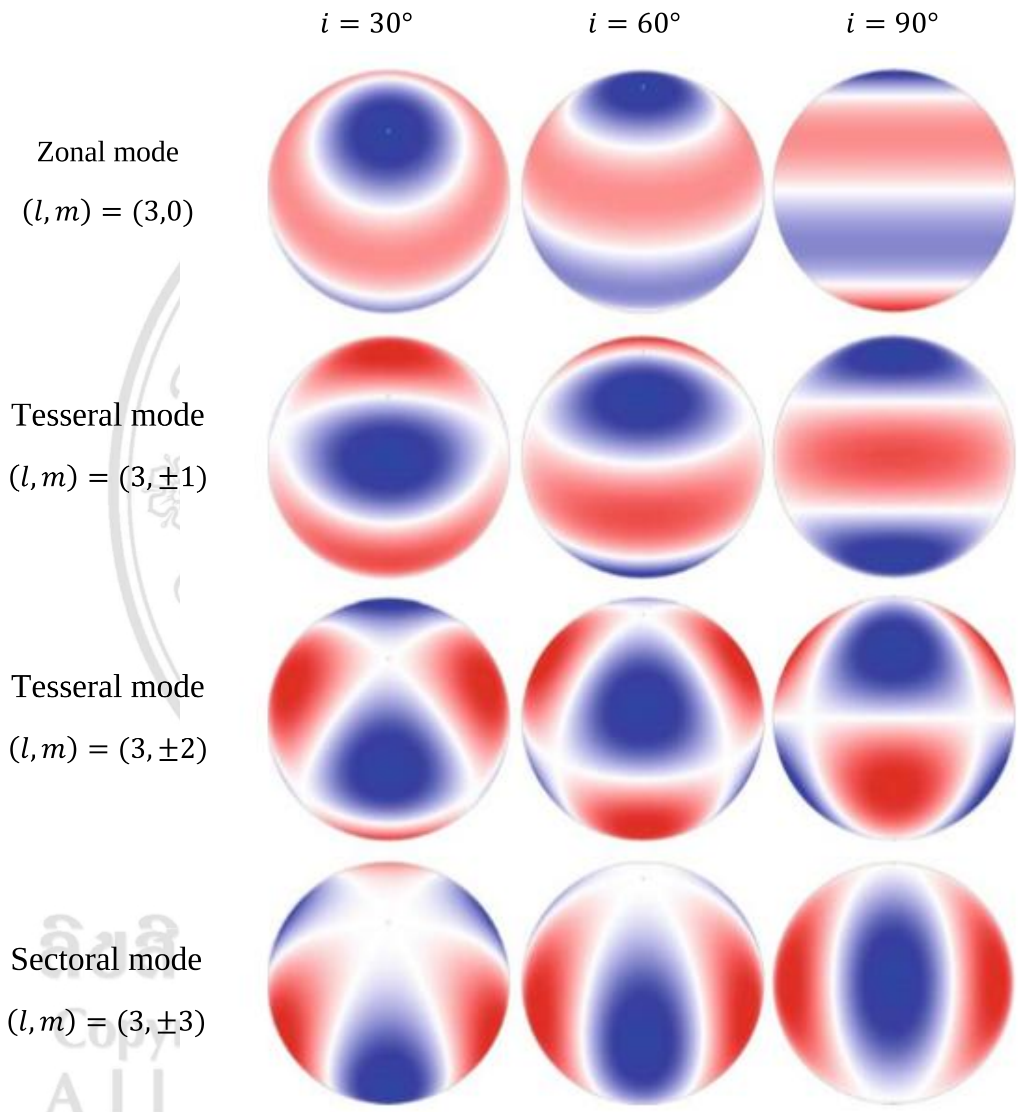

Figure 2.3: Doppler images of the surface pattern of different types of non-radial oscillation modes for different inclination angles. The mode of the star has degree $l = 3$ octupole modes. The left to the right columns show the inclination of 30°, 60° and 90°, respectively. The first row presents the axisymmetric octupole mode ($l = 3$, $m = 0$). The second row shows the tesseral mode ($l = 3$, $m = \pm 1$) with two nodes that are latitude and longitude lines. The third row shows the tesseral mode ($l = 3$, $m = \pm 2$), and the sectoral mode ($l = 3$, $m = \pm 3$) is presented in the last row. Contracting and expanding of surface areas of the star represent by red and blue (Courtesy of Aerts et al. (2010)).

Table 2.1: The properties of pulsating variables. Adapted from Cox (1980) and Carroll and Ostlie (2007).

Type	Range of Periods	Population Types	Spectral Types	Radial or Non-radial
Long Period Variables	100-700 d	I and II	M, R, N, S	R
Classical Cepheids	1-50 d	I	F6-K2	R
W Virginis Stars	2-45 d	II	F2-G6	R
γ Dor Stars	0.4-3 d	I	F0-F2	NR
RR Lyrae	1.5-24 h	II	A2-F2	R
δ Scuti Stars	1-3 h	I	A2-F5	R,NR
β Cephei Stars	3-7 h	I	B1-B2	R,NR
ZZ Ceti Stars	100-1000 s	I	A5-F5	NR

2.2.3 p Modes and g Modes

Stellar oscillations can be classified according to the nature of the dominant restoring force of the oscillation and this leads to two types of non-radial modes: p modes and g modes.

(i) The p (pressure) modes, in which the primary restoring force for a star perturbed from equilibrium is pressure. They mainly appear as radial or vertical movements at the stellar surface. These p modes are acoustic waves and they have the largest amplitude in the outer stellar layers.

(ii) The g (gravity) modes, in which the main restoring force is gravity or buoyancy and for which gas motions are primarily horizontal. Their largest amplitudes remain deep in the stellar layers, therefore, they are important to probe the stellar interiors.

The interesting characteristics of g and p modes following: (i) Number of radial nodes effect the frequencies of g and p modes; the frequencies of the g modes decrease and the frequencies of the p modes increase, while the number of radial nodes increases. (ii) The g and p modes are most sensitive to conditions in different parts of the stars; the g modes are most sensitive in the deep inner part of the stars, while the p modes are most sensitive in the outer part of the stars. (iii) There is a correlation for g modes; they are equally spaced in period and there is a correlation for p modes; they are equally spaced in frequency. Figure 2.4 illustrates the g modes in solar-like stars which are propagated below the convective zone. For stars which have convective outer zone, like the γ Dor stars, the pulsation modes are easy to observe at the surface because they propagate in the outer parts of the star. The g-modes are limited outside the convective core for high mass main-sequence stars.

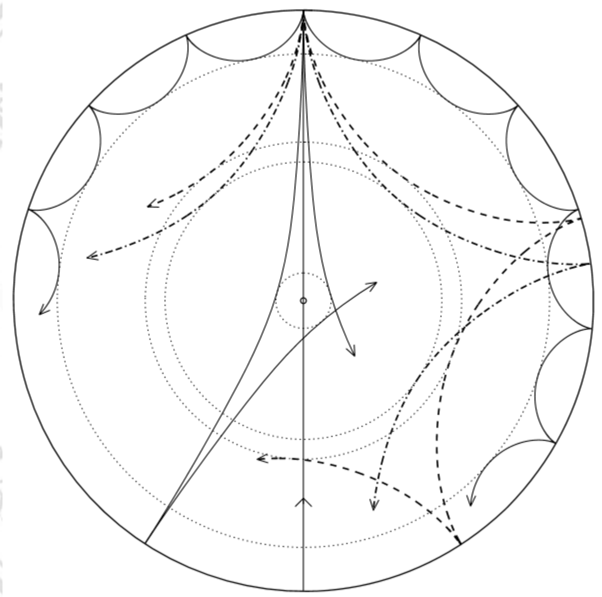

Figure 2.4: Diagram shows an interior structure of a Sun-like star and propagation of waves across its inner layers. The dotted circles represent the acoustic waves which are bend by the increase in speed of wave with depth until they arrive the inner turning point. The acoustic rays are reflected by the density gradient at the surface of the star (Courtesy of Cunha (2007)).

2.2.4 Driving Mechanism: κ-Mechanism

A star in hydrostatic equilibrium is normally quite stable against a small perturbation, but in pulsating variable stars a driving mechanism is necessary to induce and maintain their oscillations. In 1917, Eddington proposed the basic idea of an excitation mechanism that is able to produce unstable oscillations with high amplitude. This mechanism is usually called the κ-mechanism, but sometimes called *Eddington valve*. The driving force behind the κ-mechanism is the behaviour of the opacity (κ) and its gradient ($\Delta\kappa$) in the different stellar layers. κ is the absorption coefficient for radiation, it indicates the radiative opacity at any particular depth of the stellar atmosphere.

In order to excite and maintain stellar oscillations, the κ-mechanism occurs at specific layers in the stellar interior namely *partial ionization layers*, in which the opacity changes suddenly. Different regions with increased opacity occur inside stars, so called *opacity bumps* are caused by the hydrogen and helium ionization or due to absorption by excited ions of metal (Z-bump). During the pulsation cycle, a layer loses support against the star's gravity and falls down to the centre of the star. The layer contracts, heats up and becomes more opaque to radiation. The opacity κ and radiative flux F_{rad} relation (Bohm-Vitense, 1992) is given by

$$F_{rad} = -\frac{4}{3}\frac{1}{\kappa}\frac{dB}{dr}, \tag{2.24}$$

where $B = \sigma T^4/\pi$ is the Plank function. During an adiabatic contraction, F_{rad} decreases, when κ increases. In the H and He ionization layers, a certain amount of flux coming from the deeper layers cannot propagate easily to the next layer due to the increased κ. The temperature of the gas and the pressure increase causing the star to expand past its equilibrium state. In the other hand, the opacity decreases due to ionization of the gas, radiation easily flows again, causing the gas cools. In this process, radiative pressure no longer support the weight of the upper layers, so the star contracts.

Upon contraction, flux is absorbed again due to recombination of the H or He, the layers receive thermal energy, hence they behave like a heat engine. The driving layers called the partial ionization zones. Layers of the star expand and contract in pulsation cycle. Energy is lost in each pulsation cycle for most of the interior of the star, therefore, the pulsations occur almost every parts of the star. If a small part in the interior of the star receive energy, the pulsation can continue. During the contraction moment in the pulsation cycle, when a layer in the star receives heat and drives the pulsation, other layers lose heat and also damp the pulsation. The star works as a heat engine when interior region achieves in driving the oscillation. A pulsation changes thermal energy into mechanical energy called *heat-engine mechanism*.

Stochastic driving is a driving mechanism which occurs in the Sun, solar-like stars and some pulsating variables in red giant phase. In stars which the modes are very stable, the oscillations are not able to drive by the heat-engine mechanism. However, the outer convection zone of the star has enough energy to resonate its natural oscillation frequencies. Some of the stochastic signal is changed to energy driving global oscillation.

ϵ-mechanism is another theoretical driving mechanism in the stars. This mechanism is described by the energy generation rate (ϵ) in the star. Gradient of energy generation rate ($\Delta\epsilon$) could drive pulsations. Theoretically, ϵ-mechanism occurs in only fully-convective stars. It is possible in some cases of very massive and evolved stars. Currently, there is no known pulsating stars which are only driven by the ϵ mechanism.

2.2.5 The Pulsation Constant (Q)

A theoretical period-mean density relation (Ledoux and Walraven, 1958) is given by

$$Q = P\left(\frac{\rho}{\rho_\odot}\right)^{1/2}, \tag{2.25}$$

where Q is the pulsation constant as a function of P and ρ, ρ is the mean density, and P is the pulsation period. The pulsation constant value depends on type of pulsation modes. If ρ is replaced by the radius and mass relation following the equation: $\rho = 3M/4\pi R^3$, we derive:

$$Q \sim P \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R}{R_\odot}\right)^{-3/2} \tag{2.26}$$

$$\log Q \sim \log P + \frac{1}{2}\log\left(\frac{M}{M_\odot}\right) - \frac{3}{2}\log\left(\frac{R}{R_\odot}\right). \tag{2.27}$$

The following relations are introduced:

$$\frac{L}{L_\odot} = \left(\frac{R}{R_\odot}\right)^2 \left(\frac{T_{eff}}{T_{eff,\odot}}\right)^4 \tag{2.28}$$

and

$$M_{bol} - M_{bol,\odot} = -2.5\log\left(\frac{L}{L_\odot}\right). \tag{2.29}$$

If we substitute Equation 2.28, Equation 2.29, the solar values $M_{bol,\odot}$ = 4.75 and $T_{eff,\odot}$ = 5,770 K in Equation 2.27 and rearrange the terms, the pulsation constant is given by (Breger, 2000)

$$\log Q = \log P + \frac{1}{2}\log\left(\frac{M}{M_\odot}\right) - 0.3M_{bol} + 3\log T_{eff} - 12.709 \tag{2.30}$$

$$\log Q = -\log f + 3.33\log g + 0.1\log M_\odot + 10\log T_{eff} - 6.456, \tag{2.31}$$

where M is the mass and T_{eff} is the effective temperature. The Q value is constant for a specific mode. In order to obtain the value of pulsation constant, M_{bol}, T, and M of the stars must be first determined. In the case of the stars with known distances, M_{bol} is known. The model atmosphere calibrations can derive the temperature of the stars. The stellar mass can be estimated from the evolutionary models by looking at the position of the star in the H-R diagram (Breger, 2000). Comparison of the observed Q-value with the theoretical ones allows us to determine the pulsation mode. Models for δ Scuti stars have been used to derive theoretical pulsation constants and period ratios by Stellingwerf (1979), Cox et al. (1979) and Andreasen et al. (1983). Table 2.2 presents the mode of pulsation, period and pulsation constant of δ Scuti stars derived by Stellingwerf (1979).

Table 2.2: Pulsation characteristics of δ Scuti star with parameters: $M = 2M_{\odot}$, T_{eff} = 7000 K, M_{bol} = 1.21 derived by Stellingwerf (1979).

Mode of pulsation	Period (days)	Q (days)	P_i/P_{i-1}	P_i/P_0
F	0.112	0.033	-	-
1H	0.087	0.025	0.77	0.77
2H	0.071	0.021	0.82	0.63
3H	0.059	0.017	0.84	0.53
4H	0.051	0.015	0.86	0.45
5H	0.044	0.013	0.87	0.39

For the radial fundamental mode, Q equals 0.033 days. For higher radial overtones or non-radial p modes the value should be smaller whereas it should be larger in the case of non-radial g modes. Andreasen et al. (1983) have shown that pulsation constants and period ratios are sensitive to the chemical composition and can be used to estimate the degree of helium depletion in individual δ Scuti stars. The values of the pulsation constant, Q, are determined and compared with theoretical ones from pulsation models.

2.2.6 Effects of Rotation

In the equilibrium model, stars are symmetric spheres and there are no velocity fields, but it is obviously false in rotating stars. In the case of the Sun, the surface rotation depends on latitude. Any rotating coordinate system can be used to describe the occurrence of velocity fields. For large-scale velocity fields which caused by convective energy transportation in the star can affect on the pulsation modes. In principle, the observed properties of the stellar oscillations provide an important tool to probe the velocity fields. Moreover, large-scale magnetic fields can cause the star become asymmetric sphere.

From a geometrical justification and mathematics implications, the observed frequencies are influenced by rotation. The angular velocity Ω is assumed to be uniform, and oscillation with a frequency ω_0 is considered in the reference frame co-rotating with the star. A coordinate system in this frame is introduced with coordinates (r', θ', ϕ'). They associate with the coordinates (r, θ, ϕ) in an inertial reference frame as given by

$$(r', \theta', \phi') = (r, \theta, \phi - \Omega t). \tag{2.32}$$

In the case of co-rotating frame, the perturbations depend on t and ϕ_0 following this relation: $\cos(m\phi' - \omega_0 t)$. In the inertial frame, it becomes $\cos(m\phi - \omega m t)$

$$\omega_m = \omega_0 + m\Omega \tag{2.33}$$

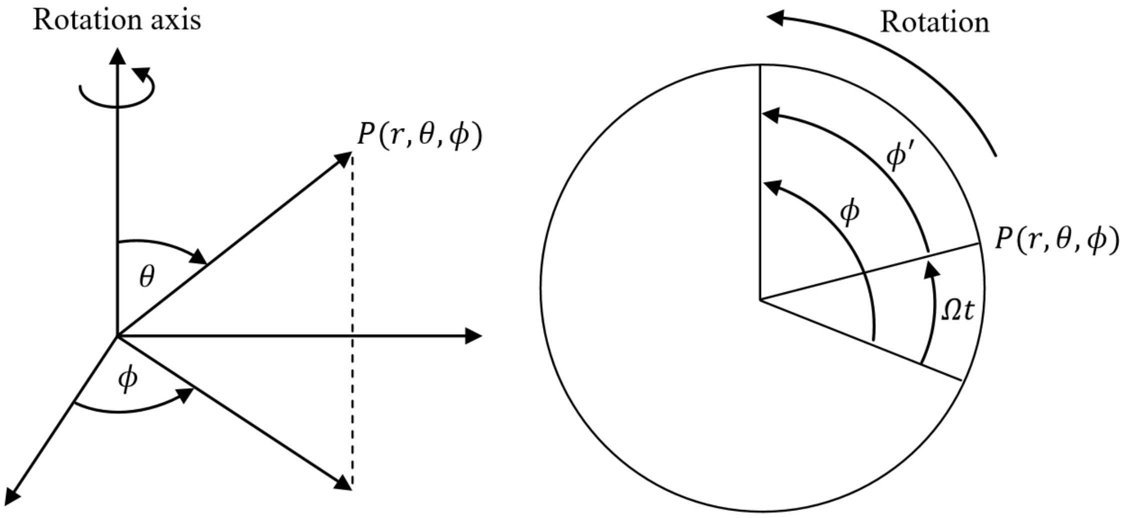

Figure 2.5: The geometry of rotational splitting. An initial spherical coordinate system (left) shows the coordinate of point $P(r, \theta, \phi)$. The axis of coordinate system coincides with rotation and axis which is also the symmetry axis for the pulsations. Top view on the north pole of rotating star with angular velocity Ω (right) shows position of P which has longitude ϕ' in the rotating system and $\phi = \phi' + \Omega t$ in the inertial system. Adapted from Cox (1984) and Aerts et al. (2010).

In the inertial frame, an observer can observe uniform splitting of the frequencies corresponding to m. This explanation is obviously incomplete in the case of uniform rotation. In the rotating frame, the effects of the Coriolis force must be considered, because it can affect the splitting of the frequency (Cowling and Newing, 1949; Ledoux, 1951). In general case, the angular velocity is a function of position as $\Omega(r, \theta)$ and the effect of the

Coriolis force is very small. If Ω is replaced by a suitable average of the angular velocity, Equation 2.33 is approximately correct.

2.3 Stellar Oscillation Across the Instability Strip

2.3.1 Solar-Like Stars

The good way to study solar-like star oscillations is study our closest star, the Sun. Its brightness variations for the strongest modes have amplitudes around 8 ppm (parts per million). Solar oscillation spectrum shows hundreds of peaks around 3 mHz which is in the range of 3-15 min. Turbulent convective motions near solar surface is the main cause of its oscillations. Such oscillations are found in all stars with outer convection layers. Theoretically, solar-like oscillations are expected to be found in the stars which located in the red edge of the classical instability strip with masses $1.6M_{\odot}$ down to the lowest mass main sequence stars. Such stochastic oscillations are not easy to detect because they have very small amplitudes, especially in the less massive stars. The velocity amplitudes were approximately predicted as a function of L/M before the first discoveries in stars other than the Sun (Kjeldsen and Bedding, 1995). For predictions of pulsations based on model of stellar outer atmosphere, the more precise relation was corrected to $(L/M)^{0.8}$ (Samadi et al., 2005).

Currently, study of helioseismology is much more advanced compared with seismic studies of solar-like oscillations in other stars. However, the astronomers continuously attempt to improve them. For asteroseismic space missions, the metal poor stars with masses lower than the Sun are interesting targets to study solar-like oscillations. Asteroseismology is very useful technique to study interior structure and estimate a very precise age of a star (Christensen-Dalsgaard, 2002). Obtaining accurate age estimation of low mass stars in a group of oldest stars in our Galaxy provide a good way to estimate the age of the Universe itself.

2.3.2 δ Scuti Stars

δ Scuti stars are short periods pulsating variables of spectral type A and F, commonly having small amplitudes. They are defined as a group of Population I pulsating stars with masses between 1.5-2.5$M_\odot$. Their position on the H-R diagram is in the lower part of the classical instability strip, where it crosses the main sequence area, so they are in shell or central hydrogen burning phase.

δ Scuti stars oscillate in both radial and nonradial oscillations, which are driven by the heat mechanism occurs in the helium second partial ionization zone. From observations, their pulsation periods are detected between 18 min to 8 hr in low order p modes and their pulsation amplitudes from mmag up to tenths of a magnitude. They are the most numerous pulsating variables among the bright stars. δ Scuti stars were known as early as 1900.

There is one special type of δ Scuti stars which is high amplitude δ Scuti stars (HADS). They are reported to be monoperiodic oscillators pulsate in radial mode. In the other hand, many nonradial oscillations have been detected in several lower amplitude δ Scuti stars. Rodríguez and López-González (2000) provided the most up-to-date catalogue of δ Scuti stars which concludes a summary of 600 group members including all the observational properties. Rodríguez and Breger (2001) also reported a extensive analysis of the characteristics of all these group members.

The δ Scuti stars show a Period-Luminosity-Colour relation as following equation (Breger and Bregman, 1975):

$$M_V = 3.052 \log P + 8.456(b-y)_0 - 3.121(1.34), \qquad (2.34)$$

therefore they can also be used as distance indicators.

Complex oscillation patterns usually present in frequency spectra of these stars, leads to a huge problem in the mode analysis. The complexity also leads to hamper detailed seismic studies. The occurrences of mixed modes are additional problem in modes identification, especially in the more evolved δ Scuti stars. Mixed mode is a mixed character of sound waves in the star itself, for example a p-mode occurs in the outer layers and a g-mode occurs in the inner part of the star. In general, such combined modes appear in stars that are undergoing hydrogen shell burning and have evolved off the main sequence.

The oscillations of F-type δ Scuti stars are strongly affected by convection in convective zones. The heat and momentum fluxes as a function of time following the formulation by Gough (1977) in calculations of δ Scuti models were reported by Houdek (2000). They found the correct position of these stars on the H-R diagram which located in the red edge region. Dupret et al. (2005) also computed the red edge of the instability strip. The results of time dependent convection treatment obtained from their study compared with those results from a frozen convection treatment indicates that their result is much better fit with observations. For special case, main sequence stars with masses around $2M_\odot$ are considered because they are in the transition limit between stars having convective outer zone ($M < 2M_\odot$) and stars having radiative outer zone ($M > 2M_\odot$). On the other hand, stars with masses ($1M_\odot < M < 2M_\odot$) develop a convective core. Study the class δ Scuti stars provides detailed knowledge in physics of the transitions from convective to radiative behaviors and blending in outer zones.

2.3.3 γ Dor Stars

γ Dor stars are a group of variables of spectral type F0-F2. It was established as a new group of Population I non-radially pulsating stars. Such stars are located near the intersection of the main sequence and the red edge of the classical instability strip. Kaye et al. (1999) defined this class of variables that have periods between 0.4 to 3 days and brightness variability amplitudes up to 0.1 mag in V filter. The early-F type star 9 Aur with amplitude around 0.1 mag was discovered before and realized that the three stars γ Doradus, HD164515, and HD96008 have similar properties (Krisciunas et al., 1993). The prototype of these early-F spectral types stars is multiperiodic γ Doradus which was first discovered by Cousins et al. (1989) and Cousins (1992) and continuously studied by Balona and Laney (1996).

New class stars with period near one day are very difficult to discover from ground based observations, thus the Hipparcos space mission is very important mission to detect these stars. About 50 stars are currently confirmed as γ Doradus stars and more than 100 stars are new candidates (e.g. De Cat et al., 2006; Cuypers et al., 2009). Individual periods of these multiperiodic stars are in the range of 0.5 and 3 d, which is longer than acoustic modes in this group of stars. Therefore, their multiperiodic variability is high order nonradial g modes.

The relationship between γ Dor and δ behaviours was studied by Handler and Shobbrook (2002). Oscillations of these two classes are very clear separation, except for the hybrid star HD 209295 which has both g and p modes. Its g modes may be tidally drive due to a member of a close eccentric binary (Handler et al., 2002). Guzik et al. (2000) first reported an excitation mechanism which located at the bottom of the convective envelope driving by convective flux. This mechanism happens in the frozen convection zone, which the disturbance to the convective flux is not considered, resulting instability strip (e.g. Warner et al., 2003).

γ Dor stars were the target group of the CoRoT space mission and they are very special target to study stellar oscillations. They pulsate in both solar-like p modes and g modes at the same time. Both types of oscillations can be found in such stars because they have very different inner regions. Study of their oscillations provides very precise information about their interior structure (e.g. Miglio et al., 2008).

2.3.4 Rapid Oscillating Ap Stars (roAp)

The rapidly oscillating Ap (roAp) stars are strongly magnetic Population I stars with A or F spectral types. They are located in the intersection between lower part of classical instability strip and main sequence region, where the δ Scuti stars are located. Atomic diffusion in these stars causes a peculiar chemical surface abundance. Kurtz (1982) discovered five stars in this class which have amplitudes of about 0.01 mag in blue filter. Currently, 40 stars are established as the class members. Multiperiodic variations which relate to low degree p modes are commonly detected in roAp stars. Frequency multiplets are detected in many modes because the magnetic fields are aligned with pulsational rotation of the star, called the oblique pulsator model (Kurtz, 1982).

The characteristics of the rapidly oscillating Ap (roAp) stars are: 1) strong abundance irregularity of element in the stars, especially rare earth elements. 2) non-uniform abundances in the stars both horizontally and vertically. 3) strong global magnetic fields with polar field intensity up to several kG (e.g. Hubrig et al., 2005; Kurtz et al., 2006). 4) high radial pulsation with radial velocity up to 5000 m/s, pulsation periods between 5.65-21.2 min and amplitudes up to 6 mmag in photometric observations (Elkin et al., 2005).

The pulsations in roAp stars are very complicated to theoretically interpret because pulsation modes are aligned with the magnetic field. Both low degree modes and extraordinary dipole modes can be found in these stars. The magnetic field and the rotation can perturb the large and small asteroseismic separations. Magnetic pressure occurs in the observable atmosphere, the modes in these stars called *magneto-acoustic*.

2.3.5 RR Lyrae

RR Lyrae stars are monoperiodic variable stars with periods between 0.2 to 1.2 d, with oscillation amplitudes from tenths of magnitude up to 2 magnitudes. They are horizontal branch stars located in the instability strip. Their lives are in the end of hydrogen core burning phase after evolved from the main sequence to the red giant phase. In 1895, Bailey discovered the first RR Lyrae stars in globular clusters. These stars must be extreme Population II stars due to their distribution, positions in the clusters and kinematic characteristics. RR Lyrae stars commonly pulsate in classical radial modes. Their monoperiodicity pulsations are not suitable for asteroseismology. The spacial characteristic of RR Lyrae stars is they can be used to determine the age and the distance of the clusters they belong to. They are chosen to be *standard candles* to study galactic evolution.

The classification of RR Lyrae stars depends on the pattern of their light curves, the amplitude and oscillation period. They are divided into several Bailey classes.

- RRa: The light curves show asymmetric shape with a fast increase and slower decrease. They pulsate in the radial fundamental mode.
- RRb: The light curves show asymmetric shape same as RRa type, but have smaller amplitude (∼0.6 mag). They pulsate in the radial fundamental mode.

- RRab: The light curves show characteristics of RRa and RRb together.
- RRc: The light curves show nearly sinusoidal shape with amplitudes $\sim$0.5 mag. They pulsate in the radial first overtone.
- RRd: They pulsate in the fundamental and first overtone radial modes, called double mode RR Lyraes.

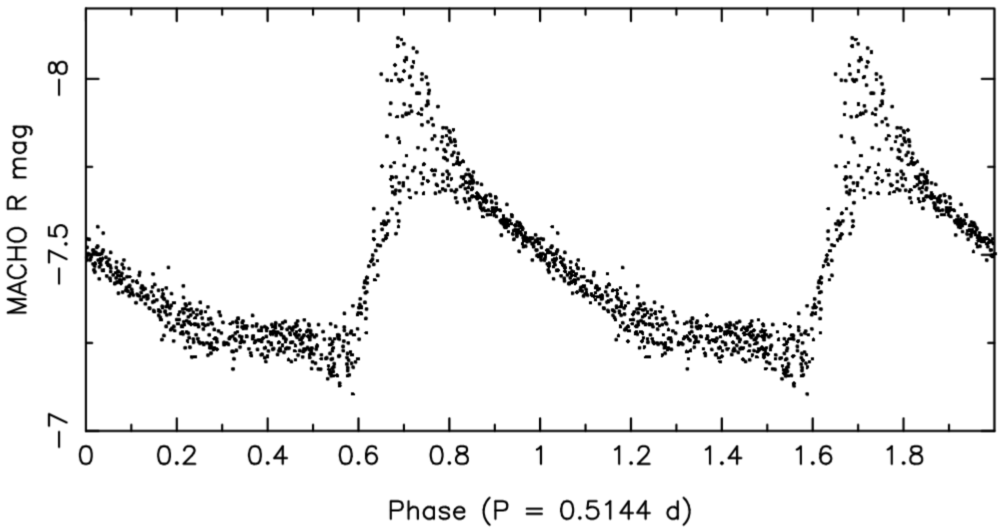

Figure 2.6: A phase curve of a Blazhko star. The dominant oscillation period of the star is 0.5144 d. Data observed by MACHO (MAssive Compact Halo Objects survey) and taken from Kurtz and Martinez (2000).

RR Lyrae stars have either crossed the strip while evolving on the horizontal branch or they immediately evolved on the horizontal branch within the instability strip after the helium flash. The heat mechanism in the He II-He III partial ionization zone is the main excitation mechanism of the RR Lyrae stars for an instability strip (e.g Stellingwerf, 1984).

Blazhko effect is a phenomena that occurs in RR Lyrae stars when pulsation amplitude is commonly one hundred times longer than the oscillation period, this effect can be found in about one forth of all RR Lyrae stars. The example of the phase curve of a Blazhko star is presented in Figure 2.6. Blažko (1907) first observed this effect for the star EW Dra. RR Lyrae itself is a Blazhko star with a pulsation period of 40.8 d, termed *Blazhko period* (e.g. Kolenberg et al., 2006). While the Blazhko cycle continually happens, the maximum brightness changes extremely and minimum brightness changes slowly. There are two theoretical interpretations for the Blazhko effect:

1) Blazhko effect is caused by the combination of the main radial mode and a nonradial oscillation mode of low degree. The Blazhko period in this model presents the beat period between the radial fundamental mode and a nonradial oscillation mode (e.g. Van Hoolst, 1995; Van Hoolst et al., 1998; Dziembowski et al., 1999).

2) Blazhko effect is caused by a magnetic field in the stars that influences the oscillations same as the oblique pulsator model in the roAp stars. The Blazhko period in this model presents the rotation period of the star (e.g. Takata and Shibahashi, 1995).

2.3.6 Cepheids

Cepheid variables are giant and supergiant stars with masses between 5-20$M_{\odot}$). They were first determined to be standard candles in 1912 (Leavitt and Pickering, 1912). Cepheids are particularly useful as standard candles because their luminosities can be computed from the Period-Luminosity Relation. Stars with birth masses larger than $\simeq 2.3M_{\odot}$ decrease in luminosity after they start burning helium in their cores, while they descend the giant branch. For stars with masses lower than $3M_{\odot}$ lie on the horizontal branch, while more massive stars show loops in the HR diagram. Stars with masses below $5M_{\odot}$ loops are too limited to bring them into the instability strip. For more massive stars, the loops move far enough so that they become pulsationally unstable Cepheids.

The special characteristic of Cepheids is the period-luminosity relation which used for distance measurements. After measuring the oscillation period of the star, the absolute magnitude and the distance to the star can be derived by using the period-luminosity relation. Thus, Cepheids are called *distance indicators*. Kurtz and Pollard (2004) presented the importance of cosmological distance scales, appropriate statistical error estimates and the derivation of the zero points.

2.4 Binary Systems

In photometric binary systems, the orbital motion of the components around the centre of mass of the system causes a periodic variation of the total system brightness. The commonly known photometric binaries are *eclipsing variables*, where the components passing in front of each other cause the brightness variations. Another class of photometric binaries are the *ellipsoidal variables*, where there are no actual eclipses. In this systems, the tidal force from one component distorts the shape of the other one, eventually become an ellipsoidal shape. The projected surface area of the distorted component varies depending on different orbital phases. A small variation in total brightness due to decrease of surface temperature at the ends of the tidal bulges.

2.4.1 Roche Model for Binary Systems

In the study of binary systems, *Roche lobe* is specifically important concept to describe gravitational equipotential around the components. When we consider a star as a single point mass, the potential energy is constant on spherical equipotential surfaces around the star and is a function only of the distance from the centre of the star. In the system of two orbiting stars, the total equipotential surfaces close to each star are almost spherical. There is a critical surface which is called *Roche lobe*.

Roche lobe model is named in honour of the nineteenth-century French mathematician Edouard Roche, who investigated the mathematics of the restricted three-body problem. The problem is considered about the motion of mass-less test particle lies near two objects or two stars. Considering orbiting masses M_1 and M_2 in the gravitational field. For this phenomenon, there are two important parts must be considered, one is the gravitational potential of the two stars and the other one is the centrifugal force acting on the mass-less test particle. There are 5 important points around the system called *Lagrangian points*. Lagrangian points are the points around the system where the effective potential gradient is zero. There are three most important Lagrangian points lie along the connected lines between two stars. The inner Lagrangian point or L_1 is the most important point which lies between two stars. The equipotential surface passes through this point and connects the gravitational spheres of influence of the two stars. The matter can flow through the L_1

Lagrangian point into the Roche lobe of the other star if one star evolves and starts to fill its Roche lobe. This is called *Roche-lobe overflow* (RLOF).

Eggleton (1983) presented the effective Roche-lobe radius R_L for binary systems. Consider the mass-ratio q and the orbital separation A between two stars. Mass M_1 is star 1 and mass M_2 is star 2, the effective Roche-lobe radius R_L is given by

$$R_L = \frac{0.49q^{-2/3}}{0.6q^{-2/3} + \ln(1 + q^{-1/3})} A, \tag{2.35}$$

where $q \equiv M_2/M_1$, radius R_L only depends on the mass-ratio q and the orbital separation A between two stars.

2.4.2 Classification of Binary Systems

Modern classification of binary stars, not only eclipsing variables, is based on the concept of *Lagrangian surfaces* and *Roche lobes*. They can be classified into three types:

1) *Detached binaries*: both components are almost spherical, they are well within their respective Roche lobes at any moment of their evolution path. They are commonly smaller than their Roche-lobe, therefore mass transfer via Roche-lobe overflow does not occur in this case. The main gravitational interaction happens through mutual tides between two stars. The mass, radii and temperature can be determined from their light curves provided there are detectable variations and from radial velocities.

2) *Semi-detached binaries*: one component fills its Roche lobe, the other is well within its Roche lobe. The former star is distorted and losing mass thought the inner Lagrangian point called *mass transfer*, the later star is almost spherical.

3) *Contact binaries*: both components are in contact due to filling their Roche lobes. There may be a common envelope of materials surrounding the two stars which are referred to an *Over-contact binary* or *Common envelope binaries*. There are L_2 and L_3 which are other two important Lagrangian points for

such systems. L_2 and L_3 located along the connected lines between the two stars but outside their orbit. Mass can flow from the binary to the circumstellar environment through L_2 or L_3 if the common envelope arrives either of these two points. A circumbinary disc may be formed from the mass flow.