CHAPTER 5

Frequency Analysis of Time-Series Data

Time-series data of pulsating stars are sets of corresponding times, in general given in Heliocentric Julian date (HJD), and monitored properties, typically light intensity (or flux) in magnitudes. Time-series can also be of other monitored, for instance moments of line profiles. There are several methods to search for frequencies in time-series data: the phase dispersion minimization (PDM) method (Stellingwerf, 1978), the Scargle periodogram (Scargle, 1981; Horne and Baliunas, 1986), the CLEAN technique (Roberts et al., 1987) and the Period98 tool (Sperl, 1998), all these are regularly used. They are all based on Fourier analysis, except for the PDM method.

5.1 Frequency Analysis Methods

The methods based on Fourier analysis are presented in this section. A Fourier analysis is very useful method for frequency analysis, it can be used for any function that has a smooth pattern. Time series with big gap are less suited for these methods, but the signal comprises of a combination of sine or cosine functions is well used in the methods. In this frequency analysis, a function of test frequencies is defined to reach the true frequency appears in the observed signal. The *periodogram* is the plot of a function of test frequencies.

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5.1.1 The Continuous Fourier Transform of an Infinite Time Series

Consider the *Fourier transform* which have finite and continuous conditions. For a function $x(t)$, its Fourier transform is presented following this equation:

$$F(\nu) \equiv \int_{-\infty}^{+\infty} x(t) \exp(2\pi i\nu t) dt. \tag{5.1}$$

There are two domains that must be considered in Fourier transform, which are domain of time and domain of frequency. The observed data in the time domain is transformed to the frequency domain using Fourier Transform. *Dirac's delta function* is the Fourier transform of the constant function 1, which is zero everywhere except at the origin:

$$\delta(\nu) \equiv \int_{-\infty}^{+\infty} \exp(2\pi i \nu t) dt, \tag{5.2}$$

which spacial properties of Dirac's delta function are given by:

$$\int_{-\infty}^{+\infty} \delta(\nu) d\nu = 1, \int_{-\infty}^{+\infty} \delta(\nu - \xi) g(\nu) d\nu = g(\xi). \tag{5.3}$$

In Fourier analysis, frequency measurement can be presented in summation of harmonic functions:

$$x(t) = \sum_{k=1}^{M} A_k \exp(2\pi i \nu_k t), \tag{5.4}$$

consider the range of frequencies between ν_1,..., ν_M and the range of amplitudes between A_1,..., A_M. Fourier Transform in term of frequency is presented by

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$$F(\nu) = \sum_{k=1}^{M} A_k \delta(\nu - \nu_k). \tag{5.5}$$

In the case of a sinusoidal function $x(t)$ with one frequency ν_1, the Fourier transform of this function is different from zero only at frequencies $\nu = \pm\nu_1$. In the case of a multiperiodic function $x(t)$ contains more than one frequency, The Fourier transform of this function is the summation of M harmonic functions with frequencies ν_1 to ν_M (Aerts et al., 2010).

5.1.2 The Continuous Fourier Transform of a Finite Time Series

For the real observations, the continuous data with infinite time series does not exist. Therefore, the case of a finite time series must be taken to account. In this case, a sample function $x(t) = A\cos[2\pi(\nu_1 t + \delta_1)]$ is substituted in the Fourier transform given in Equation 5.1. The signal is observed in finite time period from $t = 0$ until $t = T$. The Fourier transform of the sample function is given by

$$\begin{aligned}
F(\nu) &= \int_0^T x(t)\exp(2\pi i\nu t)dt \\
&= \frac{A}{2}\int_0^T \exp(2\pi i\nu t)\left\{\exp[2\pi i(\nu_1 t + \delta_1)] + \exp[-2\pi i(\nu_1 t + \delta_1)]\right\} \\
&= \frac{A}{2}\left\{\frac{\exp(2\pi i\delta_1)}{2\pi i(\nu+\nu_1)}[\exp[2\pi i(\nu+\nu_1)T - 1] + \right. \\
&\left. \frac{\exp(-2\pi i\delta_1)}{2\pi i(\nu-\nu_1)}[\exp[2\pi i(\nu-\nu_1)T - 1]\right\} \\
&= A\left\{\exp[iT\pi(\nu+\nu_1) + 2\pi i\delta_1]\frac{\sin[\frac{T}{2}2\pi(\nu+\nu_1)]}{2\pi(\nu+\nu_1)} + \right. \\
&\left. \exp[iT\pi(\nu-\nu_1) - 2\pi i\delta_1]\frac{\sin[\frac{T}{2}2\pi(\nu-\nu_1)]}{2\pi(\nu-\nu_1)}\right\}
\end{aligned} \quad . \qquad (5.6)$$

The Power periodograms $|F(\nu)|^2$ are commonly presented for periodograms. Figure 5.1 shows the function sinc(x) and its power. The pattern of power periodograms is presented by

$$\text{sinc}(x)^2 \equiv \left(\frac{\sin x}{x}\right)^2 \qquad (5.7)$$

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The peak of the sinc function or its square commonly appears exactly at ν_1 in the simple case. The two frequency peaks centred at $\pm\nu_1$ are well separated when $T >> 1/\nu_1$. The width of the sinc peak is considered in the frequency accuracy determination. The *Rayleigh criterion* is a criterion to determine accuracy of frequency. This criterion only presents a lower boundary to the accuracy of frequency (e.g. Schwarzenberg-Czerny, 2003). The true accuracy of frequency depends on the S/N ratio. In practice, noise from

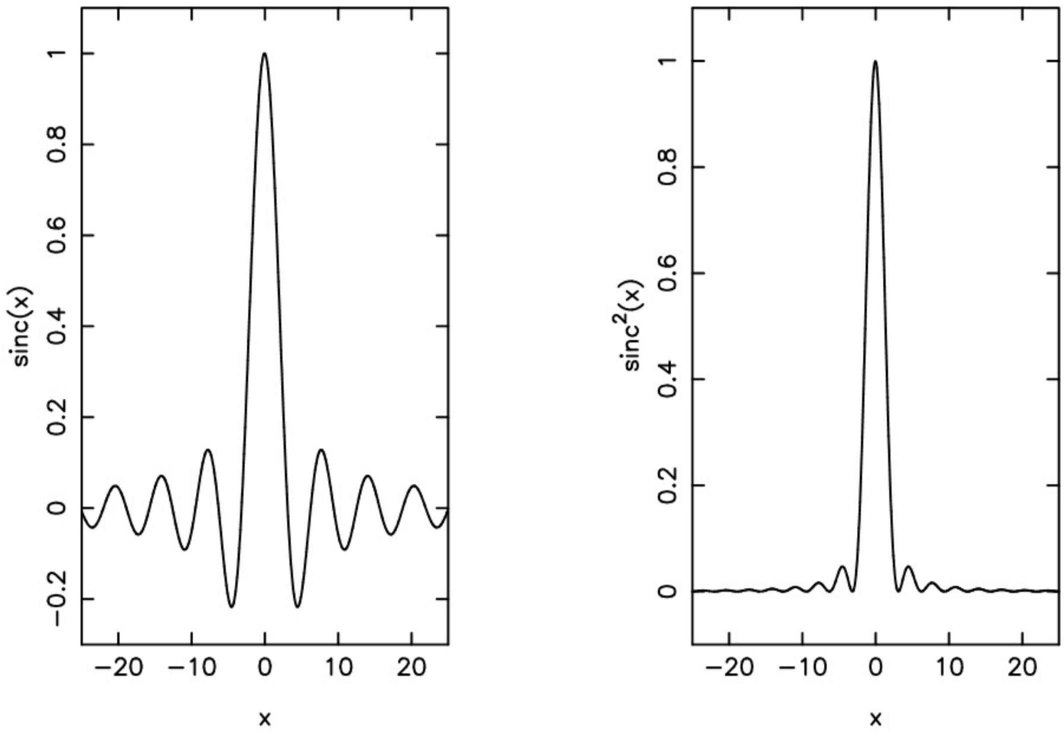

Figure 5.1: The sinc function (left panel) and its square (right panel). Courtesy of Aerts et al. (2010).

the observation can affect observed peaks in periodogram. The peaks have more complex pattern in the real data. For finite number of data points from 0 to T, frequency interference occurs duo to multiperiodic pulsation between oscillation modes.

In the case of simultaneous oscillations, the frequencies ν_1 to ν_k are well separated when $T >> 1/|\nu_i - \nu_j|$. It is used for two different peaks $i \neq j$. Frequency interference appears in the periodogram when this condition is not available. In this case, the determination of the correct frequency is more difficult. The resolving power of a periodogram was first studied by Loumos and Deeming (1978), who derived the condition $1/T < |\nu_i{-}\nu_j| < 1.5/T$, the frequencies ν_i and ν_j are well separated when this condition is complete. For the better result, they concluded that when $|\nu_i - \nu_j| > 2.5/T$, the difference between two peak frequencies and the real frequencies are few. In the real observations, the data distribute over many time intervals. In the case of *gapped data*, the resolving power of a periodogram is more sophisticated.

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5.1.3 The Discrete Fourier Transform

The data distribute over many time intervals in the real observations, therefore the discrete Fourier transform is very important. The function $x(t)$ for a discrete number of time points t_i from $i = 1$ to $i = N$ is presented. In this case, continuous Fourier transform $F(\nu)$ as a function of frequency cannot be determined. The discrete Fourier transform was introduced by Deeming (1975) following:

$$F_N(\nu) \equiv \sum_{i=1}^{N} x(t_i)\exp(2\pi i\nu t_i), \tag{5.8}$$

where N is number of measurements. Discrete Fourier transform F_N and Fourier transform F are different, but they relate with each other via the *window function* defined as

$$w_N(t) \equiv \frac{1}{N}\sum_{i=1}^{N} \delta(t - t_i). \tag{5.9}$$

The properties of the Dirac function and the window function provide a transformation of F_N to an integral form:

$$\frac{F_N}{N} = \int_{-\infty}^{+\infty} x(t)w_N(t)\exp(2\pi i\nu t)dt. \tag{5.10}$$

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The *spectral window* $W_N(\nu)$ is the discrete Fourier transform of the window function, which is given by:

$$W_N(\nu) = \frac{1}{N}\sum_{i=1}^{N} \exp(2\pi i\nu t_i). \tag{5.11}$$

The discrete Fourier transform as a function of the Fourier transform and the spectral window is given by:

$$F_N(\nu)/N = (F * W_N)(\nu). \tag{5.12}$$

$F_N(\nu)/N$ have the same property as the spectral window $W_N(\nu)$ at ν_1 if $F(\nu)$ is a δ-function at frequency ν_1. They have the same behaviour because $F_N(\nu)/N = W_N(\nu) * \delta(\nu - \nu_1) = W_N(\nu - \nu_1)$. The real frequency ν_1 can be confirmed by comparison of the $W_N(\nu)$ with $F_N(\nu)/N$ near the frequency ν_1.

When $F(\nu)$ is a sum of M δ-functions we have:

$$\begin{aligned}
\frac{F_N(\nu)}{N} &= W_N(\nu) * \sum_{k=1}^{M} \delta(\nu - \nu_k) \\
&= \sum_{k=1}^{M} W_N(\nu) * \delta(\nu - \nu_k) \\
&= \sum_{k=1}^{M} W_N(\nu - \nu_k) \\
&= \frac{1}{N} \sum_{k=1}^{M} \sum_{i=1}^{N} \exp(2\pi i(\nu - \nu_k)t_i)
\end{aligned} \tag{5.13}$$

where $F_N(\nu)/N$ is the summation of spectral windows. All spectral windows are centred on the frequencies ν_k. Sometimes, $W_N(\nu)$ differ from zero to frequencies ν, therefore the result are not equal to ν_k. In this case, maxima in the periodogram may not be the real frequencies ν_k. The time interval and noise from the observations can raise the confused maxima in the periodogram. This maxima is called *aliasing*. *Alias frequencies* are peaks due to false frequencies and the times of observations. One of disadvantage of Fourier analysis in frequency determination is alias frequencies, which occur in the spectral window.

5.1.4 The Classical Periodogram

The N measurements $(t_i, x(t_i))$ of time series are considered. The classical periodogram is written as:

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$$
\begin{aligned}
P_N(\nu) &= \frac{1}{N}|F_N(\nu)|^2 \\
&= \frac{1}{N}|\sum_{i=1}^{N} x(t_i)\exp(2\pi i\nu t_i)|^2 \\
&= \frac{1}{N}\left\{\left(\sum_{i=1}^{N} x(t_i)\sin(2\pi\nu t_i)\right)^2 + \left(\sum_{i=1}^{N} x(t_i)\cos(2\pi\nu t_i)\right)^2\right\}
\end{aligned} . \quad (5.14)
$$

If we consider a harmonic signal $x(t_i)$ in the term $x(t_i) = A\cos(2\pi\nu_1 t_i)$. In this case, the periodogram at frequency ν_1 is given by

$$
P_N(\nu_1) = \frac{1}{N}\left\{\sum_{i=1}^{N} A\cos(2\pi\nu_1 t_i)\sin(2\pi\nu_1 t_i)\right\}^2 + \frac{1}{N}\left\{\sum_{i=1}^{N} A\cos^2(2\pi\nu_1 t_i)\right\}^2 . \quad (5.15)
$$

For large N, we obtain

$$
\sum_{i=1}^{N}\cos(2\pi\nu_1 t_i)\sin(2\pi\nu_1 t_i) \approx 0, \sum_{i=1}^{N}\cos^2(2\pi\nu_1 t_i) \approx N/2, \quad (5.16)
$$

so $P_N(\nu_1) \approx A^2N/4$ for $N \to \infty$. Positive and negative terms appear for $\nu \neq \nu_1$. For such a test frequency, the overall sum will be small. The estimation $P_N(\nu_1) \approx A^2N/4$ is suited for very long time series,

$$
A(\nu) = \sqrt{\frac{4P_N(\nu)}{N}}. \quad (5.17)
$$

Amplitude $A(\nu)$ as a function of test frequency ν. The physics of the star can be interpret from a pulsation amplitude.

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5.1.5 The Lomb-Scargle Periodogram

Lomb (1976) introduced the Lomb-Scargle periodogram and then Ferraz-Mello (1981) and Scargle (1982) improved the periodogram. The formulation by Scargle (1982) is presented as following:

$$P_{LS}(\nu) = \frac{1}{2}\frac{\left\{\sum_{i=1}^{N} x(t_i)\cos[2\pi\nu(t_i-\tau)]\right\}^2}{\sum_{i=1}^{N}\cos^2[2\pi\nu(t_i-\tau)]} + \frac{\left\{\sum_{i=1}^{N} x(t_i)\sin[2\pi\nu(t_i-\tau)]\right\}^2}{\sum_{i=1}^{N}\sin^2[2\pi\nu(t_i-\tau)]}. \quad (5.18)$$

The reference epoch τ in this equation is chosen in such a way that

$$\sum_{i=1}^{N}\cos[2\pi\nu(t_i-\tau)]\sin[2\pi\nu(t_i-\tau)] = 0 \quad (5.19)$$

or

$$\tan(4\pi\nu\tau) = \frac{\sum_{i=1}^{N}\sin(4\pi\nu t_i)}{\sum_{i=1}^{N}\cos(4\pi\nu t_i)}. \quad (5.20)$$

$P_{LS}(\nu) = A^2N/4$ for extensively large N. From Equation 5.18, the amplitude spectrum based on the Lomb-Scargle periodogram is therefore defined as given by:

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$$A_{LS}(\nu) = \sqrt{\frac{4P_{LS}(\nu)}{N}} \quad (5.21)$$

The advantage of this method is that its value are unchanged whether time values t_i or $t_i + T$ is used due to the definition of τ. The Lomb-Scargle periodogram is equivalent to the variance reduction f_v, which obtained from a sinusoid fitting by least squares at test frequencies. This conclusion was derived by Horne and Baliunas (1986) and Schwarzenberg-Czerny (1997).

5.2 Program Period04

Period04 is a computer program pertaining to statistical analysis of large astronomical time series containing gaps. The advantages of the program are that it provides tools to extract the individual frequencies from the multiperiodic observed data and offers an interface to fit multiple-frequency analysis (Lenz and Breger, 2005).

The program is separated into three modules:

1) **The Time String Module**

The user can import and export the time string data in this module. The module contains tools to display table and graph. Moreover, there are tools to combine data sets, split a data set into substrings and set weights of data points.

2) **The Fit Module**

Within Fit module the user can perform least-squares fits for frequencies analysis and display phase diagram. This module contains both basic and advanced fitting techniques. It contains the tools to fit variations of amplitude or phase, or consider a periodic time shift in the observed data set. For the calculation of uncertainties of fit parameters, advanced fitting such as Monte Carlo simulations, are provided in this module.

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3) **The Fourier Module**

Within Fit module the user can extract the new frequencies from the data sets. The program performs Fourier analysis based on a discrete Fourier transform algorithm. Each time the user calculate the new frequency, they can choose calculations base on original data, residuals, or spectral window. Moreover, The module also contains tools to display table and graph.

5.2.1 The Main Window

Period04 provides the main window which is separated into three parts: the menu bar, the status bar, and a tabbed frame. The tabbed frame at the centre contains four tasks:

- **Time string** is a module for administration time string data.
- **Fit** is a module for fitting least-square fits to the data.
- **Fourier** is a module for searching a new frequency based on Fourier analysis.
- **Log** is a module which contains the recording of all actions which performed by the user.

5.2.2 The Menu Bar

For default mode, four useful menus including *File, Special, Configuration*, and *Help* are provided in the menu bar. An additional menu *Options* is available in the expert mode. The essential tasks in the menu bar of program Period04 are shown in Tables 5.1 and 5.2. Some essential tasks in the program are described following:

Nyquist Frequency

The task shows the Nyquist frequency for the selected time string data. The Nyquist frequency is estimated from the average value of gap width between close points, while long gaps are ignored.

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Creating Artificial Data

Within this task the user can easily generate artificial data having the same space as in original data set. In order to create artificial data, the user should has at least one of active frequencies in the Fit Module because the magnitude of the artificial data is created from current fit.

Calculate Noise at Frequency

Within this task the user can calculate the noise at each frequency. In the Fit Module, the frequency list contain all active frequencies.

Calculations Based On

In Fourier module the user can calculate the new frequency based on different types of data. They are "Original data", "Adjusted data", "Residuals at original" and "Residuals at adjusted". The calculation based on "Spectral window" is centered at zero frequency. The pattern of gaps in the time string data affects the structure of spectral window.

Phase plots

This task is in the Fit Module, the user can plot the phase diagram by pressing *the Phase diagram* button. In default mode, the program will use the first inactive frequency to plot the phase diagram.

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Table 5.1: The essential tasks in the menu bar which includes File and Special menus

Command	Description
File	
New Project	close the current project and create a new project
Load Project	open an existing project file in file.p04
Save Project	save the complete project into a file.p04
Save Project As	save the current project in a new name
Import	import a time string from a file and import frequencies, amplitudes, and phases from a file
Export	export time string, frequencies, and log file into a file
Manage Data	open a tool to edit project data
Expert Mode	access to additional tools in advanced mode
Quit	exit the program Perid04
Special	
Weight selection	to be used in Fourier calculations as well as least-squares calculations
Subdivide time string	divide the time string data into substrings
Combine substrings	provide a dialog to combine substrings into one set
Calculate Nyquist frequency	calculate the Nyquist frequency for current time string data
Show time structuring	show basic information of the time structurig
Adjust time string	open a dialog for adjusting zero-point variations
Delete selected points	delete all data belonging to the currently selected time string data
Select all frequencies	select all frequencies in the Fit Module
Deselect all frequencies	deselect all frequencies in the Fit Module
Clean all frequencies	clean the frequency list in the Fit Module
Calculate epochs	calculate the times of epochs for every active frequency
Recalculate residuals	recalculate the residuals using the current selection of time points and a user-defined zero poin
Predict signal	predict the magnitude of signal at a specific time
Create artificial data	open a dialog for creation of artificial data
Set alias-gap	open a dialog to set the step size for frequency adjustments
Show analytical uncertainties	display the uncertainties based on the assumption of an ideal case
Calculate noise at frequency	calculate the noise at a specific frequency
Calculate noise spectrum	provide a tool to calculate a noise spectrum

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Table 5.2: The essential tasks in the menu bar which includes Help menu

Command	Description
Help	
Period04 Help	open the link to Period04 help system
Topics	access to topics in the help system
Tutorials	open the main page for the tutorials in Period04 help system
Shortcuts	launch the lists of shortcuts that are provided in the program
Period04-Homepage	open the homepage of Period04
Report a bug	display instructions for the submission of bugs
Copyright	open the copyright notice
About	provide basic information about your Period04 version

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