CHAPTER 3

Pandiagonal Lanna Magic Squares

In this chapter, we study about how many pandiagonal Lanna Magic Squares are by using a concept of abstract algebra, a group action on subgroup of S_{16}.

Let $l =$

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

be a pandiagonal Lanna Magic Square

Since l is a pandiagonal Lanna Magic Square so, all 4 rows, each row sum is 32, all 4 columns, each sum is 32 also 2 main diagonals and 6 broken diagonals each sum is 32. And let T_1, T_2, T_3, T_4 and T_5 be transformations of pandiagonal Lanna Magic Square when

T_1: Reflection about the a, f, k, p diagonal.

T_2: Rotation through 90 ° counter-clockwise.

T_3: Putting the first column last.

T_4: Putting the first row last.

T_5: The transformation of a pandiagonal Lanna Square l into

a	d	h	e
b	c	g	f
n	o	k	j
m	p	l	i

If we consider the transformations $T_1 - T_5$ of pandiagonal Lanna Magic Squares in permutation form of subgroup of S_{16} when the number 1-16 represent the position in the square

Table 3.1: A Lanna Magic Square in permutation form

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

we get

T_1: (1)(6)(11)(16)(2 5)(3 9)(4 13)(7 10)(8 14)(12 15),

T_2: (1 4 16 13)(2 8 15 9)(3 12 14 5)(6 7 11 10),

T_3: (1 4 3 2)(5 8 7 6)(9 12 11 10)(13 16 15 14),

T_4: (1 13 5 9)(2 14 10 6)(3 15 11 7)(4 16 12 8),

T_5: (1)(7)(11)(13)(2 5 4)(3 6 8)(9 16 4)(10 12 15).

When $(i_1\, i_2 \cdots i_r)$ is a cycle of length r or an r-cycle. The notation $(i_1\, i_2 \cdots i_r)$ means that the i_1 element is replaced by the i_2, the i_2 by the i_3, $\cdots$ the i_{r-1} by i_r and the i_r by the i_1.

Example, consider one of pandiagonal Lanna Magic Squares

Table 3.2: One of pandiagonal Lanna Magic Squares

10	8	12	2
13	1	11	7
4	14	6	8
5	9	3	15

The notation T_1: (1)(6)(11)(16)(2 5)(3 9)(4 13)(7 10)(8 14)(12 15) mean that the position 1, 6, 11, 16 stay in the same position and 2 replaced by 5, 3 replaced by 9, 4 replaced by 13 and so on. Thus, applied T_1 to 3.2 we have

Table 3.3: Applied T_1 to table 3.2

10	13	4	5
8	1	14	9
12	11	6	3
2	7	8	15

It is obvious that the square 3.3 is a pandiagonal Lanna Magic Square.
For another example, consider T_3: (1 4 3 2)(5 8 7 6)(9 12 11 10)(13 16 15 14). From the notation it means that the position 1 replaced by 4, 4 replaced by 3, 3 replaced by 2 and 2 replaced by 1. The position 5 replaced by 8, 8 replaced by 7, 7 replaced by 6 and 6 replaced by 5. Namely, applied T_3 to 3.2 we get

Table 3.4: Applied T_3 to table 3.2

13	1	11	7
4	14	6	8
5	9	3	15
10	8	12	2

which is obvious to see that the square 3.4 is a pandiagonal Lanna Magic Square.

Clearly that the transformations $T_1, T_2, T_3, T_4, T_5 \in S_{16}$.

Now we will show that a pandiagonal Lanna Magic Square preserves to be a pandiagonal Lanna Magic Square after applying the transformations T_1, T_2, T_3, T_4, T_5.

Since l is a pandiagonal Lanna Magic Square so, all 4 rows, each row sum is 32. That are,

$$a + b + c + d = 32, \tag{3.1}$$

$$e + f + g + h = 32, \tag{3.2}$$

$$i + j + k + l = 32, \tag{3.3}$$

$$m + n + o + p = 32. \tag{3.4}$$

All 4 columns, each sum is 32. That are,

$$a + e + i + m = 32, \tag{3.5}$$

$$b + f + j + n = 32, \tag{3.6}$$

$$c + g + k + o = 32, \tag{3.7}$$

$$d + h + l + p = 32. \tag{3.8}$$

Moreover, 2 main diagonals each sum to 32,

$$a + f + k + p = 32, \tag{3.9}$$

$$d + g + j + m = 32. \tag{3.10}$$

Also 6 broken diagonals, each sum is 32. That are,

$$a + h + k + n = 32, \tag{3.11}$$

$$b + e + l + o = 32, \tag{3.12}$$

$$c + f + i + p = 32, \tag{3.13}$$

$$d + e + j + o = 32, \tag{3.14}$$

$$c + h + i + n = 32, \tag{3.15}$$

$$b + g + l + m = 32. \tag{3.16}$$

Consider applying transformation T_1 : reflection about the a, f, k, p diagonal, to l we have

Table 3.5: Applied T_1 to l

a	e	i	m
b	f	j	n
c	g	k	o
d	h	l	p

It is easy to see that

$$\begin{array}{llll}
a+e+i+m=32 & \text{(from 3.5)} & b+f+j+n=32 & \text{(from 3.6)} \\
c+g+k+o=32 & \text{(from 3.7)} & d+h+l+p=32 & \text{(from 3.8)} \\
a+b+c+d=32 & \text{(from 3.1)} & e+f+g+h=32 & \text{(from 3.2)} \\
i+j+k+l=32 & \text{(from 3.3)} & m+n+o+p=32 & \text{(from 3.4)} \\
a+f+k+p=32 & \text{(from 3.9)} & m+j+g+d=32 & \text{(from 3.12)} \\
a+n+k+h=32 & \text{(from 3.10)} & e+b+o+l=32 & \text{(from 3.11)} \\
i+f+c+p=32 & \text{(from 3.12)} & m+b+g+l=32 & \text{(from 3.15)} \\
i+n+c+h=32 & \text{(from 3.14)} & e+j+o+d=32 & \text{(from 3.13)}
\end{array}$$

Thus, the square 3.5 is a pandiagonal Lanna Magic Square.

In other words we can see that the transformation T_1 carries rows into columns and columns into rows so, all 4 columns and 4 rows each sum are 32. Moreover, it is clearly that all main and broken diagonals each sum are 32.

Thus, a pandiagonal Lanna Magic Square remains a pandiagonal Lanna Magic Square after applied T_1.

Consider apply transformation T_2 : Rotation through 90 ° counter-clockwise, to l we have

Table 3.6: Applied T_2 to l

d	h	l	p
c	g	k	o
b	f	j	n
a	e	i	m

It is easy to see that

$$\begin{array}{llll}
d+h+l+p=32 & \text{(from 3.8)} & c+g+k+o=32 & \text{(from 3.7)} \\
b+f+j+n=32 & \text{(from 3.6)} & a+e+i+m=32 & \text{(from 3.5)} \\
d+c+b+a=32 & \text{(from 3.1)} & h+g+f+e=32 & \text{(from 3.2)}
\end{array}$$

$l+k+j+i=32$ (from 3.3) $p+o+n+m=32$ (from 3.4)

$d+g+j+m=32$ (from 3.12) $p+k+f+a=32$ (from 3.9)

$d+o+j+e=32$ (from 3.13) $c+h+i+n=32$ (from 3.14)

$l+g+b+m=32$ (from 3.15) $p+c+f+i=32$ (from 3.12)

$l+o+b+e=32$ (from 3.11) $h+k+n+a=32$ (from 3.10)

Thus, the square 3.6 is a pandiagonal Lanna Magic Square.

In other words we can see that the transformation T_2 applied to a pandiagonal Lanna Magic Square. As in a transformation T_1, it carries rows into columns and columns into rows so, all 4 columns, 4 rows and all 2 main and 6 broken diagonals each sum are 32.

Thus, a pandiagonal Lanna Magic Square remains a pandiagonal Lanna Magic Square after applied T_2.

Consider apply transformation T_3 : Putting the first column last, to l we have

Table 3.7: Applied T_3 to l

b	c	d	a
f	g	h	e
j	k	l	i
n	o	p	m

It is easy to see that

$b+c+d+a=32$ (from 3.1) $f+g+h+e=32$ (from 3.2)

$j+k+l+i=32$ (from 3.3) $n+o+p+m=32$ (from 3.4)

$b+f+j+n=32$ (from 3.6) $c+g+k+o=32$ (from 3.7)

$d+h+l+p=32$ (from 3.8) $a+e+i+m=32$ (from 3.5)

$b+g+l+m=32$ (from 3.15) $a+h+k+n=32$ (from 3.10)

$b+e+l+o=32$ (from 3.11) $c+f+i+p=32$ (from 3.12)

$d+g+j+m=32$ (from 3.12) $a+f+k+p=32$ (from 3.9)

$d+e+j+o=32$ (from 3.13) $c+h+i+n=32$ (from 3.14)

Thus, the square 3.7 is a pandiagonal Lanna Magic Square.

In other words we can see that the transformation T_3 applied to a pandiagonal Lanna Magic Square. It carries rows into rows and columns into columns so, all 4 columns and 4 rows each sum are 32. In addition, all 2 main and 6 broken diagonals each sum still are 32.

Thus, a pandiagonal Lanna Magic Square remains a pandiagonal Lanna Magic Square after applied T_3.

Consider apply transformation T_4 : Putting the first row last, to l we have

Table 3.8: Applied T_4 to l

e	f	g	h
i	j	k	l
m	n	o	p
a	b	c	d

It is easy to see that

$$\begin{array}{llll}
e+f+g+h=32 & (\text{from } 3.2) & i+j+k+l=32 & (\text{from } 3.3) \\
m+n+o+p=32 & (\text{from } 3.4) & a+b+c+d=32 & (\text{from } 3.1) \\
e+i+m+a=32 & (\text{from } 3.5) & f+j+n+b=32 & (\text{from } 3.6) \\
g+K+o+c=32 & (\text{from } 3.7) & h+l+p+d=32 & (\text{from } 3.8) \\
e+j+o+d=32 & (\text{from } 3.13) & h+k+n+a=32 & (\text{from } 3.10) \\
e+l+o+b=32 & (\text{from } 3.11) & f+i+p+c=32 & (\text{from } 3.12) \\
g+j+m+d=32 & (\text{from } 3.12) & h+i+n+c=32 & (\text{from } 3.14) \\
g+l+m+b=32 & (\text{from } 3.15) & f+k+p+a=32 & (\text{from } 3.9)
\end{array}$$

Thus, the square 3.8 is a pandiagonal Lanna Magic Square.

In other words we can see that the transformation T_4 applied to a pandiagonal Lanna Magic Square. As in transformation T_3, it carries rows into rows and columns into columns so, it is clear that all 4 columns, 4 rows and all 2 main and 6 broken diagonals each sum are 32.

Thus, a pandiagonal Lanna Magic Square remains a pandiagonal Lanna Magic Square after applied T_4.

Before we prove that the transformation T_5 preserves pandiagonal Lanna Magic Squares' properties, we have to show some important lemmas that help us to prove it.

Lemma 3.1. *The four elements of any 2×2 subsquare of a pandiagonal Lanna Magic Square add up to 32.*

Proof. Consider a pandiagonal Lanna Magic Square l,

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

from the pandiagonal Lanna Magic Square's properties that each row, column principle diagonal and broken diagonal add up to 32.

$$\begin{aligned} \text{Consider} \quad (a+f+k+p)+(d+g+j+m)+(e+f+g+h)+ & \\ (i+j+k+l)-(a+e+i+m)-(d+h+l+p) &= 64 \\ 2(f+g+j+k) &= 64 \\ \therefore\ f+g+j+k &= 32 \end{aligned}$$

By use of transformations T_3 and T_4, this can be applied to any square of order two. □

For example, if we want to show that a, b, e, f in l sum to 32, we can first take T_3 to l 3 times and then take T_4 3 times, we will get the square

p	m	n	o
d	a	b	c
h	e	f	g
l	i	j	k

This square remains a pandiagonal Lanna Magic Square because as we show above that both T_3 and T_4 preserve pandiagonal Lanna Magic Squares' properties. So that, after we applied T_3 to l first time, the square we got also a pandiagonal Lanna Magic Square. Next, we applied T_3 again to the square we got from applying first time, it also remains a pandiagonal Lanna Magic Square and so on. After that, we can prove similar to Lemma 3.1. We will get that $a+b+e+f=32$.

Lemma 3.2. *All pandiagonal Lanna Magic Square, the sum of any element and the element that is two distant from it along a diagonal is 16. (This include main diagonals and broken diagonals.)*

Proof. Consider a pandiagonal Lanna Magic Square l,

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

from the pandiagonal Lanna Magic Square's properties that each row, column principle diagonal and broken diagonal add up to 32. Thus,

$$
\begin{aligned}
(a+b+c+d)+(i+j+k+l)+(a+e+i+m)+(c+g+k+o)+ & \\
(a+f+k+p)+(a+n+k+h)-(e+b+o+l)-(i+f+c+p)- & \\
(m+j+g+d)-(i+n+c+h) &= 64 \\
4(a+k) &= 64 \\
\therefore a+k &= 16
\end{aligned}
$$

By use of transformations T_3 and T_4, this can move position of a and k to any pair of two distant elements along diagonal. □

For example, if we want to show that $c+i=16$, we can first take T_4 to l 2 times, we will get the square

i	j	k	l
m	n	o	p
a	b	c	d
e	f	g	h

Clearly, this square remains a pandiagonal Lanna Magic Square after applied T_4. Then we can prove similar to 3.2 and we will get that $c+i=16$.

Now we can prove that the transformation T_5 preserves a pandiagonal Lanna Magic Square properties.

Consider apply transformation T_5 to l we have

Table 3.9: Applied T_5 to l

a	d	h	e
b	c	g	f
n	o	k	j
m	p	l	i

According to l; a pandiagonal Lanna Magic Square,

$l =$

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

From l we have,

$$a+f+k+p=32 \quad c+h+i+n=32$$
$$e+b+o+l=32 \quad d+g+j+m=32$$

and by lemma 3.1 we have,

$$a+d+h+e=32 \quad b+c+g+f=32$$
$$n+o+k+j=32 \quad m+p+l+i=32$$
$$a+b+n+m=32 \quad d+c+o+p=32$$
$$h+g+k+l=32 \quad e+f+j+i=32$$

and by lemma 3.2 we have,

$$a+k=16 \quad c+i=16 \quad \text{so,} \quad a+c+k+i=32$$
$$e+o=16 \quad g+m=16 \quad \text{so,} \quad e+g+o+m=32$$
$$b+l=16 \quad d+j=16 \quad \text{so,} \quad b+d+j+l=32$$
$$h+n=16 \quad f+p=16 \quad \text{so,} \quad h+f+n+p=32$$

Thus, the square 3.9 is a pandiagonal Lanna Magic Square or we can say that a pandiagonal Lanna Magic Square remains a pandiagonal Lanna Magic Square after applied T_5.

Thereby, a pandiagonal Lanna Magic Square remains a pandiagonal Lanna Magic Square after applied transformations T_1, T_2, T_3, T_4, T_5.

Let $G =< T_1, T_2, T_3, T_4, T_5 >$ be a subgroup of S_{16} generated by $T_1, T_2, T_3, T_4,$ and T_5 and e be the identity of G and let L be a set of all pandiagonal Lanna Magic Squares.

Define $F : G \times L \to L$ by $(g, l) \mapsto gl = l \circ g, \forall g \in G, l \in L$

Since, $T_i l = l \circ T_i \in L \ \forall i \in \{1, 2, 3, 4, 5\}$ and $G =< T_1, T_2, T_3, T_4, T_5 >$, we have F is an action of a group G.

Theorem 3.3. *All pandiagonal Lanna Magic Squares can be derived from a single one (the Lanna magic square) by successive transformations of* $T_1, T_2, T_3, T_4,$ *and* T_5.

Proof. Let l be any pandiagonal Lanna Magic Square,

$l=$

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

.

By use of transformation T_3 and T_4, the number 1 in any pandiagonal Lanna Magic Squares can be brought into the position f, the second row and the second column. So, there is no loss of generality in taking $f = 1$ and by Lemma 3.2, we have $p = 15$.

a	b	c	d
e	1	g	h
i	j	k	l
m	n	o	15

From $f = 1$ then g and h can be at most 13 and 14. Thus, $g + h \leq 27$ and we get $e \geq 4$. Similarly, e and h can be at most 13 and 14 so, $e + h \leq 27$ we derive $g \geq 4$. In the same way, we also acquire $h \geq 4$, $b \geq 4$, $j \geq 4$, $n \geq 4$, $a \geq 4$, $k \geq 4$.

By Lemma 3.1, we know that $b + c + f + g = 32, e + f + i + j = 32$. Thus, b and g can be at most 13 and 14 so, $b + g \leq 27$ and we get $c \geq 4$. Similarly, we also get $i \geq 4$. Thus, 2 and 3 can only occur in d, l, m and o.

If 2 occurs in o, it can be brought into l by taking T_5 2 times. If it occurs in l, it can be brought into d by taking T_3 2 times and if it occurs in m, it can be brought into d by taking T_1. Hence we take $d = 2$ then we also get $j = 14$. Now we have the square

a	b	c	2
e	1	g	h
i	14	k	l
m	n	o	15

If 3 occurs in l, it can be brought into o by taking T_5, if it occurs in p, it can be brought into o by taking T_3 3 times and If it occurs in m, it can be brought into o by taking T_3 2 times. So, one can take $o = 3$ and then $e = 13$.

a	b	c	2
13	1	g	h
i	14	k	l
m	n	3	15

By Lemma 3.1, we also get $i = 4$ and $c = 12$.

a	b	12	2
13	1	g	h
4	14	k	l
m	n	3	15

Now we have

$$\begin{aligned}
m &= 32 - e - i - a &&= 32 - 13 - 4 - a &&= 15 - a \\
g &= 32 - d - j - m &&= 32 - 2 - 14 - 15 + a &&= a + 1 \\
n &= 32 - m - o - p &&= 32 - 15 + a - 3 - 15 &&= a - 1 \\
b &= 32 - a - c - d &&= 32 - 12 - 2 - a &&= 18 - a \\
k &= 32 - a - f - p &&= 32 - a - 1 - 15 &&= 16 - a \\
l &= 32 - i - j - k &&= 32 - 4 - 14 - 16 + a &&= a - 2 \\
h &= 32 - e - f - g &&= 32 - 13 - 1 - a - 1 &&= 17 - a
\end{aligned}$$

a	$18-a$	12	2
13	1	$a+1$	$17-a$
4	14	$16-a$	$a-2$
$15-a$	$a-1$	3	15

So, we have to find values of a such that $15-a, a+1, a-1, 18-a, 16-l, a-2$ and $17-a$ are 5, 6, 7, 8, 9, 10, 11 in some order. For instance, if $a = 5$, consider $h = a-2 = 3$ which already exists at o. So, $a \neq 5$ and by doing all substitution, a can be only 7 or 10.

For $a = 7$, we get

7	11	12	2
13	1	8	10
4	14	9	5
8	6	3	15

and for $a = 10$, we get exactly the same as the Lanna Magic Square

10	8	12	2
13	1	11	7
4	14	6	8
5	9	3	15

The square $a = 7$ can be the same as the square $a = 10$ by applying $T_2^2 T_1 T_2^3 T_5 T_3^2$.

Thus, all pandiagonal Lanna Magic Squares can be obtained from the Lanna Magic Square by applying application T_1, T_2, T_3, T_4 and T_5 . □

From Theorem 3.3, we can say that the group $G = < T_1, T_2, T_3, T_4, T_5 >$ acts on L is transitive.

The proof of next theorem is similar to Theorem 4 in [9]. However, for the sake of completeness, we describe below.

Theorem 3.4. *The order of subgroup of S_{16} generated by T_1, T_2, T_3, T_4, T_5 is 384.*

Proof. If T_X and T_Y are two transformations, denote by $T_X T_Y$. The transformation effected by first applying T_Y and then T_X.

Consider

$$\begin{array}{llll} (1) & T_2T_1 = T_1T_2^3 & (2) & T_3T_1 = T_1T_4 \\ (3) & T_4T_1 = T_1T_3 & (4) & T_3T_2 = T_2T_4 \\ (5) & T_4T_2 = T_2T_3^3 & (6) & T_4T_3 = T_3T_4 \\ (7) & T_5T_1 = T_1T_2^2T_3T_4T_5 & (8) & T_5^2T_1 = T_2^2T_3T_4T_5^2 \\ (9) & T_5T_2 = T_2^3T_3T_4T_5 & (10) & T_5^2T_2 = T_1T_2T_3T_4T_5^2 \end{array}$$

(11) $T_5T_3 = T_3^3T_5^2$ (12) $T_5^2T_3 = T_2^2T_3T_4^2T_5$

(13) $T_5T_4 = T_1T_2^2T_4T_5$ (14) $T_5^2T_4 = T_1T_2^2T_4T_5$

(15) $T_1^2 = T_2^4 = T_3^4 = T_4^4 = T_5^3$ which is identical transformation.

By inspection from (1)-(15), any product of T_1, T_2, T_3, T_4, T_5 all can get T_1 to the left, then T_2 next to T_1, then T_3, T_4 and T_5 is on the right. So, any product of T_1, T_2, T_3, T_4, T_5 equal to the form $T_1^\alpha T_2^\beta T_3^\gamma T_4^\delta T_5^\epsilon$. For the transformation in Theorem 3.3 $T_2^2T_1T_2^3T_5T_3^2 = T_1T_2^3T_3^2T_4^2T_5$.

It is clear that T_2, T_3, T_4 are independent. Moreover, T_5 or T_5^2 is not the product of T_1, T_2, T_3, T_4 since all of T_1, T_2, T_3, T_4 carry rows into rows, columns into columns, columns into rows or rows into columns which cannot yield neither T_5 nor T_5^2. Besides the transformation T_2, T_3, T_4 preserve the orientation so, T_1 is not a product of T_2, T_3, T_4.

Therefore, $T_1^\alpha T_2^\beta T_3^\gamma T_4^\delta T_5^\epsilon = T_1^aT_2^bT_3^cT_4^dT_5^e$ if and only if $\alpha = a$, $\beta = b$, $\gamma = c$, $\delta = d$ and $\epsilon = e$.

Thus, the order of subgroup generated by $T_1, T_2, T_3, T_4\, T_5$ is $2\times4\times4\times4\times3 = 384$. □

Theorem 3.5. *There are 384 pandiagonal Lanna Magic Squares.*

Proof. Since G acts on L, consider for each $l \in L$ there is only the identity e of G such that $el = l$. So, $G_l =< e >$.

From Theorem 3.3 the group G acts on L is transitive so, by Theorem 2.6.5 for $l \in L$ we have that the orbit of l is L.

From Theorem 2.6.3 the cardinal number of the orbit $l \in L$ is $[G : G_l]$ and Theorem 3.4 $|G| = 384$ we have $|L| = |l| = [G : G_l] = [G :< e >] = |G| = 384$.

Hence, there are 384 pandiagonal Lanna Magic Squares. □