

การหาสูตรเรซิดิวของฟังก์ชันเศษส่วนเชิงซ้อน

(Residue Formula of Complex Rational Function)

การหาสูตรเรซิดิวของฟังก์ชันเศษส่วนเชิงซ้อน ก็เพื่อจะนำไปหาค่าเรซิดิว แล้วจึงนำเอาค่าเรซิดิวที่หาได้ไปใช้ในการหาอินทิกรัลของ $\oint_C f(z)dz$ เมื่อ C เป็น คอนทัวร์ปิดเชิงเดียว และ $f(z)$ เป็นฟังก์ชันวิเคราะห์ ภายในและบน C ยกเว้นที่ จุดเอกเทศ (Isolated singularity) z_0 ภายใน C จากนิยาม 3. บทที่ 4 อนุกรมของโลรองต์ $f(z)$ รอบจุด $z = z_0$ คือ

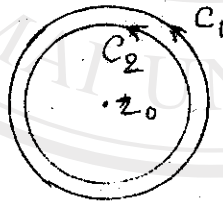
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_{-n} (z-z_0)^{-n}$$

เมื่อ $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-z_0)^{n+1}} dw$ และ $n = 0, 1, 2, 3, \dots$

$b_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-z_0)^{-n+1}} dw$ และ $n = 1, 2, 3, 4, \dots$

ถ้า C_2 เป็นวงกลมภายใน C_1 ที่มีจุดศูนย์กลางที่จุด z_0 จากทฤษฎีที่ 2

บทที่ 3 จะได้ว่า $\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$



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จากสูตร b_{-n} เมื่อ $n = 1$ จะได้

$$b_{-1} = \frac{1}{2\pi i} \oint_{C_2} f(w)dw = \frac{1}{2\pi i} \oint_{C_2} f(z)dz$$

ดังนั้น $\oint_{C_2} f(z)dz = 2\pi i b_{-1}$

จะเห็นว่าอินทิกรัลของ $f(z)$ บน C ซึ่งล้อมรอบจุดเอกเทศที่ $z = z_0$ มีความสัมพันธ์กับสัมประสิทธิ์ b_{-1} ของการกระจายอนุกรมของโลรองต์ของ $f(z)$ รอบจุด $z = z_0$ ดังนั้นต่อไปนี้จะศึกษาวิธีการหาสัมประสิทธิ์ b_{-1} เพื่อนำไปหา $\oint_C f(z) dz$

นิยาม 1 ถ้า $f(z)$ มีจุดเอกเทศ (Isolated singularity) ที่จุด $z = z_0$ และ $f(z)$ กระจายเป็นอนุกรมของโลรองต์ในวงแหวน $r_2 < |z - z_0| < r_1$ ได้ว่า

$$f(z) = \dots + \frac{b_{-3}}{(z-z_0)^3} + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

แล้วจะเรียกสัมประสิทธิ์ b_{-1} ของ $\frac{1}{z-z_0}$ ว่าเป็นค่าเรซิดิว (Residue) ของ $f(z)$ ณ จุด z_0 เขียนแทนด้วย $\text{Res}(f(z), z_0)$ หรือ $\text{Res}(z_0)$

ตัวอย่าง 1 จงหาค่าเรซิดิวของ $f(z) = z.e^{3/z}$ ที่จุด $z = 0$ เมื่อ C เป็นวงกลม

$|z| = 4$

วิธีทำ กระจายอนุกรมโลรองต์ของฟังก์ชัน $z.e^{3/z}$ รอบจุด $z = 0$ จะได้ดังนี้

$$\begin{aligned} z.e^{3/z} &= z \left\{ 1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 \frac{1}{2!} + \left(\frac{3}{z}\right)^3 \frac{1}{3!} + \dots \right\} \\ &= z + 3 + \frac{3^2}{2!}z + \frac{3^3}{3!}z^2 + \dots \end{aligned}$$

จากนิยาม 1 จะได้ว่า $\text{Res}(f(z), 0) = \frac{9}{2}$

ทฤษฎีที่ 1 ถ้า $f(z)$ มีโพลเชิงเดียว (Simple pole) ที่จุด $z = z_0$ จะได้ว่า

$$\text{Res}(f(z), z_0) = \lim_{z \rightarrow z_0} (z-z_0)f(z) \dots \dots \dots (5.1)$$

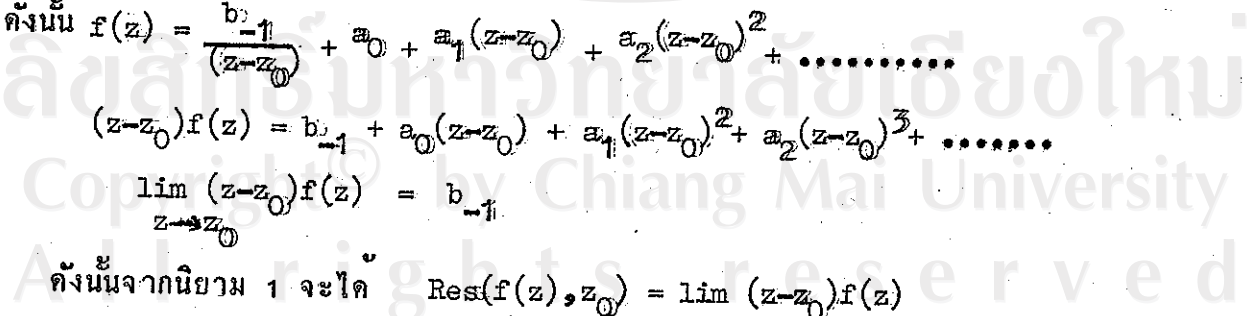
พิสูจน์ กำหนดให้ $f(z)$ มีโพลเชิงเดียวที่จุด $z = z_0$

ดังนั้น $f(z) = \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$

$$(z-z_0)f(z) = b_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \dots$$

$$\lim_{z \rightarrow z_0} (z-z_0)f(z) = b_{-1}$$

ดังนั้นจากนิยาม 1 จะได้ $\text{Res}(f(z), z_0) = \lim_{z \rightarrow z_0} (z-z_0)f(z)$



ตัวอย่าง 2 จงหาค่าเรซิดิวของ $f(z) = \frac{e^z}{z(z+1)}$ ที่จุด $z = 0$ และ $z = -1$

วิธีทำ $f(z) = \frac{e^z}{z(z+1)}$ มีโพลเชิงเดี่ยวที่จุด $z = 0$ และ $z = -1$

จากทฤษฎีที่ 1 จะได้ว่า

$$\begin{aligned} \text{Res}(0) &= \lim_{z \rightarrow 0} z f(z) \\ &= \lim_{z \rightarrow 0} \frac{e^z}{z+1} \\ &= 1 \end{aligned}$$

และ

$$\begin{aligned} \text{Res}(-1) &= \lim_{z \rightarrow -1} (z+1)f(z) \\ &= \lim_{z \rightarrow -1} e^z \\ &= e^{-1} \\ &= \frac{1}{e} \end{aligned}$$

ทฤษฎีที่ 2 ถ้า $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์โดยที่ $p(z)$, $q(z)$ สอดคล้องกับเงื่อนไข $q(z_0) = 0$, $q'(z_0) \neq 0$ และ $p(z_0) \neq 0$ แล้วฟังก์ชัน $f(z)$ จะมีโพลเชิงเดี่ยว (simple pole) ที่จุด $z = z_0$ และค่าเรซิดิว (Residue) ของ $f(z)$ ที่จุด z_0 จะมีค่าเท่ากับ $\frac{p(z_0)}{q'(z_0)}$ (5.2)

พิสูจน์ กำหนดให้ $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ที่จุด $z = z_0$ ซึ่ง $p(z_0) \neq 0$ และ $q(z_0) = 0$ แต่ $q'(z_0) \neq 0$

แสดงว่า $q(z)$ มีซีโรอันดับที่ 1 ที่ $z = z_0$
 ดังนั้นจากทฤษฎีที่ 4 บทที่ 4 จะได้ว่า $f(z)$ มีโพลเชิงเดี่ยว

จาก $f(z) = \frac{p(z)}{q(z)}$

กระจายฟังก์ชันวิเคราะห์ $p(z)$ และ $q(z)$ โดยใช้ออนุกรมของเทย์เลอร์

$$f(z) = \frac{p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \dots}{q'(z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \dots}$$

เอา $(z-z_0)$ คูณตลอด

$$(z-z_0)f(z) = \frac{p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \dots}{q'(z_0) + \frac{q''(z_0)}{2!}(z-z_0)^2 + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \dots}$$

$$\therefore \lim_{z \rightarrow z_0} (z-z_0)f(z) = \frac{p(z_0)}{q'(z_0)}$$

จาก(5.1) จะได้ $\text{Res}(f(z), z_0) = \lim_{z \rightarrow z_0} (z-z_0)f(z) = \frac{p(z_0)}{q'(z_0)}$

ทฤษฎีที่ 3 ถ้า $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ โดยที่ $p(z)$, $q(z)$ สอดคล้องกับเงื่อนไข $q(z_0) = q'(z_0) = 0$ แต่ $q''(z_0) \neq 0$ และ $p(z_0) \neq 0$ แล้ว $f(z)$ จะมีโพลอันดับที่สองที่จุด $z = z_0$ และค่าเรซิดิวของ $f(z)$ คือ

$$\text{Res}(f(z), z_0) = \left[\frac{z!}{q''(z_0)} \right]^2 \left| \begin{array}{cc} \frac{q''(z_0)}{2!} & p(z_0) \\ \frac{q'''(z_0)}{3!} & p'(z_0) \end{array} \right| \dots \dots \dots (5.3)$$

พิสูจน์ กำหนดให้ $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็น

ฟังก์ชันวิเคราะห์ที่จุด $z = z_0$ ซึ่ง $p(z_0) \neq 0$ และ $q(z_0) = q'(z_0) = 0$ แต่ $q''(z_0) \neq 0$ แสดงว่า $q(z)$ มีซีโรอันดับที่สองที่ $z = z_0$

ดังนั้นจากทฤษฎีที่ 4 บทที่ 4 จะได้ว่า $f(z)$ มีโพลอันดับที่สอง

กระจายฟังก์ชันวิเคราะห์ $p(z)$ และ $q(z)$ โดยใช้อนุกรมของเทย์เลอร์

$$f(z) = \frac{p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \dots}{\frac{q''(z_0)}{2!}(z-z_0)^2 + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \frac{q^{IV}(z_0)}{4!}(z-z_0)^4 + \dots}$$

ในการกระจาย $f(z)$ แบบอนุกรมของโลรองต์นั้น ถ้า $f(z)$ มีโพลอันดับที่สอง

จะได้ว่า $f(z) = \frac{p(z)}{q(z)} = \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$

จาก $f(z) = \frac{p(z)}{q(z)}$

$p(z) = q(z)f(z)$

$$p(z) = \left[\frac{q''(z_0)}{2!}(z-z_0)^2 + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \dots \right] \left[\frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots \right]$$

$$p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \frac{p'''(z_0)}{3!}(z-z_0)^3 + \dots = \left[\frac{q''(z_0)}{2!}(z-z_0)^2 + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \frac{q^{IV}(z_0)}{4!}(z-z_0)^4 + \dots \right] \left[\frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots \right]$$

นั่นคือ $p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \frac{p'''(z_0)}{3!}(z-z_0)^3 + \dots$

$$= \frac{b_{-2}q''(z_0)}{2!} + \left[\frac{b_{-2}q'''(z_0)}{3!} + \frac{b_{-1}q^{IV}(z_0)}{4!} \right] (z-z_0) + \dots$$

โดยการเทียบสัมประสิทธิ์จะได้ว่า

$$\frac{b_{-2}q''(z_0)}{2!} = p(z_0)$$

$$b_{-2} = \frac{2!}{q''(z_0)} p(z_0)$$

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และ $b_{-2} \frac{q'''(z_0)}{3!} + b_{-1} \frac{q''(z_0)}{2!} = p'(z_0)$

$$\begin{aligned} b_{-1} \frac{q''(z_0)}{2!} &= p'(z_0) - b_{-2} \frac{q'''(z_0)}{3!} \\ &= p'(z_0) - \frac{2!}{q''(z_0)} \times \frac{q'''(z_0)}{3!} p(z_0) \\ b_{-1} &= \frac{2!}{q''(z_0)} p'(z_0) - \left[\frac{2!}{q''(z_0)} \right] \frac{q'''(z_0)}{3!} p(z_0) \end{aligned}$$

$$b_{-1} = \left[\frac{2!}{q''(z_0)} \right]^2 \begin{vmatrix} \frac{q''(z_0)}{2!} & p(z_0) \\ \frac{q'''(z_0)}{3!} & p'(z_0) \end{vmatrix}$$

เพราะฉะนั้น $\text{Res}(f(z), z_0)$

$$= \left[\frac{2!}{q''(z_0)} \right]^2 \begin{vmatrix} \frac{q''(z_0)}{2!} & p(z_0) \\ \frac{q'''(z_0)}{3!} & p'(z_0) \end{vmatrix}$$

ทฤษฎีที่ 4 ถ้า $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ โดยที่ $p(z)$, $q(z)$ สอดคล้องกับเงื่อนไข $q(z_0) = q'(z_0) = q''(z_0) = 0$ แต่ $q'''(z_0) \neq 0$ และ $p(z_0) \neq 0$ แล้ว $f(z)$ จะมีโพลอันดับที่สาม ที่จุด $z = z_0$ และค่าเรซิดิวของ $f(z)$ คือ

$$\text{Res}(f(z), z_0) = \left[\frac{3!}{q'''(z_0)} \right]^3 \begin{vmatrix} \frac{q'''(z_0)}{3!} & 0 & p(z_0) \\ \frac{q''(z_0)}{4!} & \frac{q'''(z_0)}{3!} & p'(z_0) \\ \frac{q'(z_0)}{5!} & \frac{q''(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{vmatrix} \dots\dots\dots (5.4)$$

พิสูจน์ กำหนดให้ $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ ที่จุด $z = z_0$ ซึ่ง $p(z_0) \neq 0$ และ $q(z_0) = q'(z_0) = q''(z_0) = 0$ แต่ $q'''(z_0) \neq 0$

แสดงว่า $q(z)$ มีซีโรอันดับที่สามที่จุด $z = z_0$
 ดังนั้นจากทฤษฎีที่ 4 บทที่ 4 จะได้ว่า $f(z)$ มีโพลอันดับที่สาม
 กระจายฟังก์ชันวิเคราะห์ $p(z)$ และ $q(z)$ โดยใช้สูตรของเทย์เลอร์

$$f(z) = \frac{p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{q'''(z_0)}{3!}(z-z_0)^3 + \frac{q^{IV}(z_0)}{4!}(z-z_0)^4 + \frac{q^V(z_0)}{5!}(z-z_0)^5 + \dots}{\dots}$$

ในการกระจาย $f(z)$ แบบอนุกรมของโลรองต์นั้น ถ้า $f(z)$ มีโพลอันดับที่สาม
 จะได้ว่า $f(z) = \frac{p(z)}{q(z)} = \frac{b_{-3}}{(z-z_0)^3} + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$

จาก $f(z) = \frac{p(z)}{q(z)}$

$p(z) = q(z)f(z)$

$$p(z) = \left\{ \frac{q'''(z_0)}{3!}(z-z_0)^3 + \frac{q^{IV}(z_0)}{4!}(z-z_0)^4 + \frac{q^V(z_0)}{5!}(z-z_0)^5 + \dots \right\} \left\{ \frac{b_{-3}}{(z-z_0)^3} + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots \right\}$$

$$p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \frac{p'''(z_0)}{3!}(z-z_0)^3 + \dots$$

$$= b_{-3} \frac{q'''(z_0)}{3!} + \left[b_{-3} \frac{q^{IV}(z_0)}{4!} + b_{-2} \frac{q'''(z_0)}{3!} \right] (z-z_0) + \left[b_{-3} \frac{q^V(z_0)}{5!} + b_{-2} \frac{q^{IV}(z_0)}{4!} + b_{-1} \frac{q'''(z_0)}{3!} \right] (z-z_0)^2 + \dots$$

โดยการเทียบสัมประสิทธิ์จะได้อ่า

$$b_{-3} \frac{q'''(z_0)}{3!} = p(z_0)$$

$$b_{-3} = \frac{3!}{q'''(z_0)} \times p(z_0)$$

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$$b_{-3} \frac{q^{IV}(z_0)}{4!} + b_{-2} \frac{q^{III}(z_0)}{3!} = p''(z_0)$$

$$b_{-2} \frac{q^{III}(z_0)}{3!} = p''(z_0) - b_{-3} \frac{q^{IV}(z_0)}{4!}$$

$$= p''(z_0) - \frac{3!}{4!} p(z_0) \frac{q^{IV}(z_0)}{q^{III}(z_0)}$$

$$b_{-2} = \frac{3! p''(z_0)}{q^{III}(z_0)} - \frac{(3!)^2 p(z_0)}{4!} \times \frac{q^{IV}(z_0)}{[q^{III}(z_0)]^2}$$

$$b_{-3} \frac{q^{IV}(z_0)}{5!} + b_{-2} \frac{q^{IV}(z_0)}{4!} + b_{-1} \frac{q^{III}(z_0)}{3!} = \frac{p''(z_0)}{2!}$$

$$b_{-1} \frac{q^{III}(z_0)}{3!} = \frac{p''(z_0)}{2!} - b_{-3} \frac{q^{IV}(z_0)}{5!} - b_{-2} \frac{q^{IV}(z_0)}{4!}$$

$$b_{-1} \frac{q^{III}(z_0)}{3!} = \frac{p''(z_0)}{2!} - \frac{3!}{q^{III}(z_0)} \times \frac{q^{IV}(z_0) p(z_0)}{5!} - \left[\frac{3!}{q^{III}(z_0)} \times p''(z_0) \right] \left[\frac{3!}{q^{III}(z_0)} \right]^2 \frac{q^{IV}(z_0) p(z_0)}{4!} \frac{q^{IV}(z_0)}{4!}$$

$$= \frac{p''(z_0)}{2!} - \frac{3!}{q^{III}(z_0)} \times \frac{q^{IV}(z_0) p(z_0)}{5!} - \frac{3!}{q^{III}(z_0)} \times \frac{q^{IV}(z_0)}{4!} \times p''(z_0) +$$

$$\left[\frac{3!}{q^{III}(z_0)} \right]^2 \times \left[\frac{q^{IV}(z_0)}{4!} \right]^2 \times p(z_0)$$

$$= \left[\frac{3!}{q^{III}(z_0)} \right]^2 \left\{ \frac{q^{III}(z_0)}{3!} \times \frac{p''(z_0)}{2!} - \frac{q^{III}(z_0) q^{IV}(z_0) p(z_0)}{3! \times 5!} - \right.$$

$$\left. \frac{q^{III}(z_0) q^{IV}(z_0) p'(z_0)}{3! \times 4!} + \left[\frac{q^{IV}(z_0)}{4!} \right]^2 p(z_0) \right\}$$

$$= \left[\frac{3!}{q^{III}(z_0)} \right]^2 \left\{ \frac{q^{III}(z_0)}{3!} \times \frac{p''(z_0)}{2!} - \frac{q^{III}(z_0) q^{IV}(z_0) p(z_0)}{3! \times 4!} + \left[\frac{q^{IV}(z_0)}{4!} \right]^2 p(z_0) \right.$$

$$\left. - \frac{q^{III}(z_0) q^{IV}(z_0) p(z_0)}{3! \times 5!} \right\}$$

$$= \left[\frac{3!}{q'''(z_0)} \right]^2 \left\{ \frac{q'''(z_0)}{3!} \begin{vmatrix} \frac{q'''(z_0)}{3!} & p''(z_0) \\ \frac{q^{IV}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{vmatrix} - \frac{q^{IV}(z_0)}{4!} \begin{vmatrix} 0 & p(z_0) \\ \frac{q^{IV}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{vmatrix} + \right.$$

$$\left. \frac{q^{IV}(z_0)}{5!} \begin{vmatrix} 0 & p(z_0) \\ \frac{q'''(z_0)}{3!} & p'(z_0) \end{vmatrix} \right\}$$

$$= \left[\frac{3!}{q'''(z_0)} \right]^2 \begin{vmatrix} \frac{q'''(z_0)}{3!} & 0 & p(z_0) \\ \frac{q^{IV}(z_0)}{4!} & \frac{q'''(z_0)}{3!} & p''(z_0) \\ \frac{q^{IV}(z_0)}{5!} & \frac{q^{IV}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{vmatrix}$$

$$b_{-1} = \left[\frac{3!}{q'''(z_0)} \right]^3 \begin{vmatrix} \frac{q'''(z_0)}{3!} & 0 & p(z_0) \\ \frac{q^{IV}(z_0)}{4!} & \frac{q'''(z_0)}{3!} & p''(z_0) \\ \frac{q^{IV}(z_0)}{5!} & \frac{q^{IV}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{vmatrix}$$

$$\therefore b_{-1} = \text{Res}(f(z), z_0) = \left[\frac{3!}{q'''(z_0)} \right]^3 \begin{vmatrix} \frac{q'''(z_0)}{3!} & 0 & p(z_0) \\ \frac{q^{IV}(z_0)}{4!} & \frac{q'''(z_0)}{3!} & p''(z_0) \\ \frac{q^{IV}(z_0)}{5!} & \frac{q^{IV}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{vmatrix}$$

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ทฤษฎีที่ 5 ถ้า $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ โดยที่ $p(z)$, $q(z)$ สอดคล้องกับเงื่อนไข $q(z_0) = q'(z_0) = q''(z_0) = q'''(z_0) = 0$ แต่ $q^{(IV)}(z_0) \neq 0$ และ $p(z_0) \neq 0$ แล้ว $f(z)$ จะมีโพลอันดับที่สี่ที่จุด $z = z_0$ และค่าเรซิดิวของ $f(z)$ คือ

$$\text{Res}(f(z), z_0) = \left[\frac{4!}{q^{(IV)}(z_0)} \right] \begin{vmatrix} \frac{q^{(IV)}(z_0)}{4!} & 0 & 0 & p(z_0) \\ \frac{q^{(V)}(z_0)}{5!} & \frac{q^{(IV)}(z_0)}{4!} & 0 & p'(z_0) \\ \frac{q^{(VI)}(z_0)}{6!} & \frac{q^{(V)}(z_0)}{5!} & \frac{q^{(IV)}(z_0)}{4!} & \frac{p''(z_0)}{2!} \\ \frac{q^{(VII)}(z_0)}{7!} & \frac{q^{(VI)}(z_0)}{6!} & \frac{q^{(V)}(z_0)}{5!} & \frac{p'''(z_0)}{3!} \end{vmatrix} \dots (5.5)$$

พิสูจน์ กำหนดให้ $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ ที่จุด $z = z_0$ ซึ่ง $p(z_0) \neq 0$ และ $q(z_0) = q'(z_0) = q''(z_0) = q'''(z_0) = 0$ แต่ $q^{(IV)}(z_0) \neq 0$

แสดงว่า $q(z)$ มีซีโรอันดับที่สี่ ที่จุด $z = z_0$ ดังนั้นจากทฤษฎีที่ 4 บทที่ 4 จะได้ว่า $f(z)$ มีโพลอันดับที่สี่ กระจายฟังก์ชันวิเคราะห์ $p(z)$ และ $q(z)$ โดยใช้อนุกรมของเทย์เลอร์

$$f(z) = \frac{p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \frac{p'''(z_0)}{3!}(z-z_0)^3 + \dots}{\frac{q^{(IV)}(z_0)}{4!}(z-z_0)^4 + \frac{q^{(V)}(z_0)}{5!}(z-z_0)^5 + \frac{q^{(VI)}(z_0)}{6!}(z-z_0)^6 + \frac{q^{(VII)}(z_0)}{7!}(z-z_0)^7 + \dots}$$

ในการกระจาย $f(z)$ แบบอนุกรมของโลตรง ดังนั้นถ้า $f(z)$ มีโพลอันดับที่สี่จะได้ว่า

$$f(z) = \frac{p(z)}{q(z)} = \frac{b_{-4}}{(z-z_0)^4} + \frac{b_{-3}}{(z-z_0)^3} + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

จาก $f(z) = \frac{p(z)}{q(z)}$

$$p(z) = \left[\frac{q^{IV}(z_0)}{4!}(z-z_0)^4 + \frac{q^{V}(z_0)}{5!}(z-z_0)^5 + \frac{q^{VI}(z_0)}{6!}(z-z_0)^6 + \frac{q^{VII}(z_0)}{7!}(z-z_0)^7 + \dots \right]$$

$$\dots \left[\frac{b_{-4}}{(z-z_0)^4} + \frac{b_{-3}}{(z-z_0)^3} + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots \right]$$

$$p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \frac{p'''(z_0)}{3!}(z-z_0)^3 + \dots$$

$$= b_{-4} \frac{q^{IV}(z_0)}{4!} + \left[b_{-4} \frac{q^{V}(z_0)}{5!} + b_{-3} \frac{q^{IV}(z_0)}{4!} \right] (z-z_0) + \left[b_{-4} \frac{q^{VI}(z_0)}{6!} + \right.$$

$$b_{-3} \frac{q^{V}(z_0)}{5!} + b_{-2} \frac{q^{IV}(z_0)}{4!} \left. \right] (z-z_0)^2 + \left[b_{-4} \frac{q^{VII}(z_0)}{7!} + b_{-3} \frac{q^{VI}(z_0)}{6!} + \right.$$

$$b_{-2} \frac{q^{V}(z_0)}{5!} + b_{-1} \frac{q^{IV}(z_0)}{4!} \left. \right] (z-z_0)^3 + \dots$$

โดยการเทียบสัมประสิทธิ์จะได้

$$b_{-4} \frac{q^{IV}(z_0)}{4!} = p(z_0)$$

$$b_{-4} = \frac{4!}{q^{IV}(z_0)} \times p(z_0)$$

$$b_{-4} \frac{q^{V}(z_0)}{5!} + b_{-3} \frac{q^{IV}(z_0)}{4!} = p'(z_0)$$

$$b_{-3} \frac{q^{IV}(z_0)}{4!} = p'(z_0) - b_{-4} \frac{q^{V}(z_0)}{5!}$$

$$= p'(z_0) - \frac{4!}{q^{IV}(z_0)} \times \frac{q^{V}(z_0)}{5!} \times p(z_0)$$

$$b_{-3} = \frac{4!}{q^{IV}(z_0)} \times p'(z_0) - \left[\frac{4!}{q^{IV}(z_0)} \right]^2 \times \frac{q^{V}(z_0)}{5!} p(z_0)$$

$$b_{-4} \frac{q^{VI}(z_0)}{6!} + b_{-3} \frac{q^{V}(z_0)}{5!} + b_{-2} \frac{q^{IV}(z_0)}{4!} = \frac{p''(z_0)}{2!}$$

$$\begin{aligned}
 b_{-2} \frac{q^{\text{IV}}(z_0)}{4!} &= \frac{p''(z_0)}{2!} - b_{-4} \frac{q^{\text{VI}}(z_0)}{6!} - b_{-3} \frac{q^{\text{V}}(z_0)}{5!} \\
 &= \frac{p''(z_0)}{2!} - \frac{4!}{q^{\text{IV}}(z_0) \cdot 6!} \times \frac{q^{\text{VI}}(z_0) p(z_0)}{p(z_0)} - \left\{ \frac{4!}{q^{\text{IV}}(z_0)} \times \frac{p'(z_0)}{p(z_0)} \right\} \\
 &\quad - \left\{ \frac{4!}{q^{\text{IV}}(z_0)} \times \frac{q^{\text{V}}(z_0) p(z_0)}{5!} \right\} \frac{q^{\text{V}}(z_0)}{5!} \\
 &= \frac{p''(z_0)}{2!} - \frac{4!}{q^{\text{IV}}(z_0) \cdot 6!} \times \frac{q^{\text{VI}}(z_0) p(z_0)}{p(z_0)} - \frac{4!}{q^{\text{IV}}(z_0)} \times \frac{q^{\text{V}}(z_0) p'(z_0)}{5!} + \\
 &\quad \left[\frac{4!}{q^{\text{IV}}(z_0)} \times \frac{q^{\text{V}}(z_0)}{5!} \right]^2 p(z_0)
 \end{aligned}$$

$$\begin{aligned}
 b_{-2} &= \frac{4!}{q^{\text{IV}}(z_0)} \times \frac{p''(z_0)}{2!} - \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^2 \times \frac{q^{\text{VI}}(z_0) p(z_0)}{6!} - \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^2 \times \frac{q^{\text{V}}(z_0) p'(z_0)}{5!} \\
 &\quad + \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^3 \times \frac{q^{\text{V}}(z_0)}{5!} p(z_0)
 \end{aligned}$$

$$b_{-4} \frac{q^{\text{VII}}(z_0)}{7!} + b_{-3} \frac{q^{\text{VI}}(z_0)}{6!} + b_{-2} \frac{q^{\text{V}}(z_0)}{5!} + b_{-1} \frac{q^{\text{IV}}(z_0)}{4!} = \frac{p'''(z_0)}{3!}$$

$$\begin{aligned}
 b_{-1} \frac{q^{\text{IV}}(z_0)}{4!} &= \frac{p'''(z_0)}{3!} - b_{-4} \frac{q^{\text{VII}}(z_0)}{7!} - b_{-3} \frac{q^{\text{VI}}(z_0)}{6!} - b_{-2} \frac{q^{\text{V}}(z_0)}{5!} \\
 &= \frac{p'''(z_0)}{3!} - \frac{4!}{3!} \times \frac{q^{\text{VII}}(z_0) p(z_0)}{7!} - \frac{4!}{q^{\text{IV}}(z_0) \cdot 6!} \times \frac{q^{\text{VI}}(z_0) p'(z_0)}{p(z_0)} \\
 &\quad + \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^2 \times \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) p(z_0)}{5! \cdot 6!} - \frac{4!}{q^{\text{IV}}(z_0) \cdot 5!} \times \frac{q^{\text{V}}(z_0) p''(z_0)}{2!} \\
 &\quad + \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^2 \times \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) p(z_0)}{5! \cdot 6!} + \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^2 \times \frac{q^{\text{V}}(z_0)}{5!} p'(z_0)
 \end{aligned}$$

$$\begin{aligned}
 & - \left[\frac{4!}{q^{\text{IV}}(z_0)} \times \frac{q^{\text{V}}(z_0)}{5!} \right]^3 p(z_0) \\
 b) \frac{q^{\text{IV}}(z_0)}{4!} &= \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^3 \left\{ \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^3 \frac{p'''(z_0)}{3!} - \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^2 \frac{q^{\text{VII}}(z_0)}{7!} \times p(z_0) \right. \\
 & - \left. \left[\frac{q^{\text{IV}}(z_0)}{4!} \right] \frac{q^{\text{VI}}(z_0)}{6!} \times p'(z_0) + \frac{q^{\text{IV}}(z_0) q^{\text{V}}(z_0) q^{\text{VI}}(z_0)}{4! \times 5! \times 6!} p(z_0) \right. \\
 & - \left. \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^2 \frac{q^{\text{V}}(z_0) p''(z_0)}{5! \times 2!} + \frac{q^{\text{IV}}(z_0) q^{\text{V}}(z_0) q^{\text{VI}}(z_0)}{4! \times 5! \times 6!} p(z_0) \right. \\
 & \left. + \frac{q^{\text{IV}}(z_0)}{4!} \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 p''(z_0) - \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 p(z_0) \right\} \\
 & - \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^4 \left\{ \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^3 \frac{p'''(z_0)}{3!} - \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^2 \frac{q^{\text{V}}(z_0) p''(z_0)}{5! \times 2!} \right. \\
 & \left. + \frac{q^{\text{IV}}(z_0)}{4!} \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 p''(z_0) - \left[\frac{q^{\text{IV}}(z_0)}{4!} \right] \frac{q^{\text{VI}}(z_0) p'(z_0)}{6!} \right\} - \\
 & \left\{ \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 p(z_0) - \frac{q^{\text{IV}}(z_0) q^{\text{V}}(z_0) q^{\text{VI}}(z_0)}{4! \times 5! \times 6!} p(z_0) \right\} + \\
 & \frac{q^{\text{IV}}(z_0) q^{\text{V}}(z_0) q^{\text{VI}}(z_0)}{4! \times 5! \times 6!} p(z_0) - \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^2 \frac{q^{\text{VII}}(z_0)}{7!} p(z_0) \\
 & = \left[\frac{q^{\text{IV}}(z_0)}{4!} \right] \left\{ \left[\frac{q^{\text{IV}}(z_0)}{4!} \right]^2 \frac{p'''(z_0)}{3!} - \frac{q^{\text{IV}}(z_0) q^{\text{V}}(z_0) p''(z_0)}{4! \times 5! \times 2!} \right\} - \\
 & \frac{q^{\text{V}}(z_0)}{5!} \left\{ - \frac{q^{\text{V}}(z_0) p'(z_0)}{5!} \right\} + \frac{q^{\text{VI}}(z_0)}{6!} \left\{ - \frac{q^{\text{IV}}(z_0) p''(z_0)}{4!} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{q^{\text{V}}(z_0)}{5!} \left\{ 0 + \left[\frac{q^{\text{V}}(z_0)}{5!} \right] p(z_0) - \frac{q^{\text{IV}}(z_0) q^{\text{VI}}(z_0) p(z_0)}{4! 6!} \right\} + \\
 & \frac{q^{\text{VI}}(z_0)}{6!} \left\{ 0 + \frac{q^{\text{IV}}(z_0) q^{\text{V}}(z_0) p(z_0) + 0}{4! 5!} \right\} - \frac{q^{\text{VII}}(z_0)}{7!} \left\{ 0 - \left[\frac{q^{\text{IV}}(z_0)}{4!} \right] p(z_0) + 0 \right\} \\
 & = \frac{q^{\text{IV}}(z_0)}{4!} \left\{ \frac{q^{\text{IV}}(z_0)}{4!} \left| \begin{array}{cc} \frac{q^{\text{IV}}(z_0)}{4!} & \frac{p''(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p'''(z_0)}{3!} \end{array} \right| - \frac{q^{\text{V}}(z_0)}{5!} \left| \begin{array}{cc} 0 & \frac{p'(z_0)}{1!} \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p'''(z_0)}{3!} \end{array} \right| \right. \\
 & + \frac{q^{\text{VI}}(z_0)}{6!} \left. \left| \begin{array}{cc} 0 & \frac{p''(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{array} \right| - \frac{q^{\text{V}}(z_0)}{5!} \left\{ 0 \left| \begin{array}{cc} \frac{q^{\text{V}}(z_0)}{4!} & \frac{p''(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p'''(z_0)}{3!} \end{array} \right| \right. \right. \\
 & \left. \left. - \frac{q^{\text{V}}(z_0)}{5!} \left| \begin{array}{cc} 0 & p(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p'''(z_0)}{3!} \end{array} \right| + \frac{q^{\text{VI}}(z_0)}{6!} \left| \begin{array}{cc} 0 & p(z_0) \\ \frac{q^{\text{IV}}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{array} \right| \right\} \right. \\
 & + \frac{q^{\text{VI}}(z_0)}{6!} \left. \left\{ 0 \left| \begin{array}{cc} 0 & \frac{p''(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p'''(z_0)}{3!} \end{array} \right| - \frac{q^{\text{IV}}(z_0)}{4!} \left| \begin{array}{cc} 0 & p(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p'''(z_0)}{3!} \end{array} \right| \right. \right. \\
 & + \frac{q^{\text{VI}}(z_0)}{6!} \left. \left\{ 0 \left| \begin{array}{cc} 0 & p(z_0) \\ 0 & \frac{p'(z_0)}{1!} \end{array} \right| - \frac{q^{\text{VII}}(z_0)}{7!} \left\{ 0 \left| \begin{array}{cc} 0 & \frac{p''(z_0)}{2!} \\ \frac{q^{\text{IV}}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{array} \right| \right. \right. \\
 & \left. \left. - \frac{q^{\text{IV}}(z_0)}{4!} \left| \begin{array}{cc} 0 & p(z_0) \\ \frac{q^{\text{IV}}(z_0)}{4!} & \frac{p''(z_0)}{2!} \end{array} \right| + \frac{q^{\text{V}}(z_0)}{5!} \left| \begin{array}{cc} 0 & p(z_0) \\ 0 & \frac{p''(z_0)}{2!} \end{array} \right| \right\} \right. \\
 & \left. \left. \left. \right. \right. \right.
 \end{aligned}$$

$$= \frac{q^{\text{IV}}(z_0)}{4!} \begin{vmatrix} \frac{q^{\text{V}}(z_0)}{4!} & 0 & p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{q^{\text{IV}}(z_0)}{4!} & p''(z_0) \\ \frac{q^{\text{V}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & p'''(z_0) \end{vmatrix} - \frac{q^{\text{V}}(z_0)}{5!} \begin{vmatrix} 0 & 0 & p(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{q^{\text{IV}}(z_0)}{4!} & p''(z_0) \\ \frac{q^{\text{V}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & p'''(z_0) \end{vmatrix}$$

$$+ \frac{q^{\text{VI}}(z_0)}{6!} \begin{vmatrix} 0 & 0 & p(z_0) \\ \frac{q^{\text{V}}(z_0)}{4!} & 0 & p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & p'''(z_0) \end{vmatrix} - \frac{q^{\text{VII}}(z_0)}{7!} \begin{vmatrix} 0 & 0 & p(z_0) \\ \frac{q^{\text{V}}(z_0)}{4!} & 0 & p''(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{q^{\text{V}}(z_0)}{4!} & p'''(z_0) \end{vmatrix}$$

$$= \begin{vmatrix} \frac{q^{\text{V}}(z_0)}{4!} & 0 & 0 & p(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{q^{\text{IV}}(z_0)}{4!} & 0 & p''(z_0) \\ \frac{q^{\text{V}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & \frac{q^{\text{IV}}(z_0)}{4!} & p'''(z_0) \\ \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{V}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & p'''(z_0) \end{vmatrix}$$

$$\begin{vmatrix} \frac{q^{\text{IV}}(z_0)}{4!} & 0 & 0 & p(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{q^{\text{IV}}(z_0)}{4!} & 0 & p''(z_0) \\ \frac{q^{\text{V}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & \frac{q^{\text{IV}}(z_0)}{4!} & p'''(z_0) \\ \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{V}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & p'''(z_0) \end{vmatrix}$$

$$\therefore \text{Res}(f(z), z_0) = b_{-1} = \left[\frac{4!}{q^{\text{IV}}(z_0)} \right]^{4!}$$

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ทฤษฎีที่ 6 ถ้า $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ โดยที่ $p(z)$, $q(z)$ สอดคล้องกับเงื่อนไข $q(z_0) = q'(z_0) = q''(z_0) = q'''(z_0) = q^{IV}(z_0) = 0$ แต่ $q^{V}(z_0) \neq 0$ และ $p(z_0) \neq 0$ แล้ว $f(z)$ จะมีโพลอันดับที่ห้าที่จุด $z = z_0$ และค่าเรซิดิวคือ

$$\text{Res}(f(z), z_0) = \left[\frac{5!}{q^{V}(z_0)} \right] \begin{vmatrix} \frac{q^{V}(z_0)}{5!} & 0 & 0 & 0 & p(z_0) \\ \frac{q^{VI}(z_0)}{6!} & \frac{q^{V}(z_0)}{5!} & 0 & 0 & p'(z_0) \\ \frac{q^{VII}(z_0)}{7!} & \frac{q^{VI}(z_0)}{6!} & \frac{q^{V}(z_0)}{5!} & 0 & p''(z_0) \\ \frac{q^{VIII}(z_0)}{8!} & \frac{q^{VII}(z_0)}{7!} & \frac{q^{VI}(z_0)}{6!} & \frac{q^{V}(z_0)}{5!} & p'''(z_0) \\ \frac{q^{IX}(z_0)}{9!} & \frac{q^{VIII}(z_0)}{8!} & \frac{q^{VII}(z_0)}{7!} & \frac{q^{VI}(z_0)}{6!} & \frac{q^{V}(z_0)}{5!} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad (5.7)$$

พิสูจน์ กำหนดให้ $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ ที่จุด $z = z_0$ ซึ่ง $p(z_0) \neq 0$ และ $q(z_0) = q'(z_0) = q''(z_0) = q'''(z_0) = q^{IV}(z_0) = 0$ แต่ $q^{V}(z_0) \neq 0$

แสดงว่า $q(z)$ มีซีโรอันดับที่ห้า ที่จุด $z = z_0$

ดังนั้นจากทฤษฎี 4 ขทที่ 4 จะได้ว่า $f(z)$ มีโพลอันดับที่ห้า

กระจายฟังก์ชันวิเคราะห์ $p(z)$ และ $q(z)$ โดยใช้นุกรมของเทย์เลอร์

$$f(z) = \frac{p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \frac{p'''(z_0)}{3!}(z-z_0)^3 + \dots}{\frac{q^{V}(z_0)}{5!}(z-z_0)^5 + \frac{q^{VI}(z_0)}{6!}(z-z_0)^6 + \frac{q^{VII}(z_0)}{7!}(z-z_0)^7 + \dots}$$

ในการกระจาย $f(z)$ แบบอนุกรมของโลรองต์นั้น ถ้า $f(z)$ มีโพลอันดับที่ห้า จะได้ว่า

$$f(z) = \frac{p(z)}{q(z)} = \frac{b_{-5}}{(z-z_0)^5} + \frac{b_{-4}}{(z-z_0)^4} + \frac{b_{-3}}{(z-z_0)^3} + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + \dots$$

จาก $f(z) = \frac{p(z)}{q(z)}$

$$p(z) = q(z)f(z)$$

$$p(z) = \left[\frac{q^{(V)}(z_0)}{5!}(z-z_0)^5 + \frac{q^{(VI)}(z_0)}{6!}(z-z_0)^6 + \frac{q^{(VII)}(z_0)}{7!}(z-z_0)^7 + \frac{q^{(VIII)}(z_0)}{8!}(z-z_0)^8 + \dots \right.$$

$$\left. \frac{q^{(IX)}(z_0)}{9!}(z-z_0)^9 + \dots \right] \left[\frac{b_{-5}}{(z-z_0)^5} + \frac{b_{-4}}{(z-z_0)^4} + \frac{b_{-3}}{(z-z_0)^3} + \frac{b_{-2}}{(z-z_0)^2} + \right.$$

$$\left. \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots \right]$$

$$p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)}{2!}(z-z_0)^2 + \frac{p'''(z_0)}{3!}(z-z_0)^3 + \frac{p^{(IV)}(z_0)}{4!}(z-z_0)^4 + \dots$$

$$= b_{-5} \frac{q^{(V)}(z_0)}{5!} + \left[b_{-5} \frac{q^{(VI)}(z_0)}{6!} + b_{-4} \frac{q^{(V)}(z_0)}{5!} \right] (z-z_0) + \left[b_{-5} \frac{q^{(VII)}(z_0)}{7!} + \right.$$

$$b_{-4} \frac{q^{(VI)}(z_0)}{6!} + b_{-3} \frac{q^{(V)}(z_0)}{5!} \left. \right] (z-z_0)^2 + \left[b_{-5} \frac{q^{(VIII)}(z_0)}{8!} + b_{-4} \frac{q^{(VII)}(z_0)}{7!} + \right.$$

$$b_{-3} \frac{q^{(VI)}(z_0)}{6!} + b_{-2} \frac{q^{(V)}(z_0)}{5!} \left. \right] (z-z_0)^3 + \left[b_{-5} \frac{q^{(IX)}(z_0)}{9!} + b_{-4} \frac{q^{(VIII)}(z_0)}{8!} + \right.$$

$$b_{-3} \frac{q^{(VII)}(z_0)}{7!} + b_{-2} \frac{q^{(VI)}(z_0)}{6!} + b_{-1} \frac{q^{(V)}(z_0)}{5!} \left. \right] (z-z_0)^4 + \dots$$

โดยการเทียบสัมประสิทธิ์ จะได้ว่า

$$b_{-5} \frac{q^{(V)}(z_0)}{5!} = p(z_0)$$

$$b_{-5} = \frac{5!}{q^{(V)}(z_0)} p(z_0)$$

$$b_{-5} \frac{q^{(VI)}(z_0)}{6!} + b_{-4} \frac{q^{(V)}(z_0)}{5!} = p'(z_0)$$

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$$b_{-4} \frac{q^{\nabla}(z_0)}{5!} = p''(z_0) - b_{-5} \frac{q^{\nabla\nabla}(z_0)}{6!}$$

$$= p''(z_0) - \frac{5!}{q^{\nabla}(z_0)} \times \frac{q^{\nabla\nabla}(z_0) p(z_0)}{6!}$$

$$b_{-4} = \frac{5!}{q^{\nabla}(z_0)} \times p''(z_0) - \left[\frac{5!}{q^{\nabla}(z_0)} \right] \frac{q^{\nabla\nabla}(z_0) p(z_0)}{6!}$$

$$b_{-5} \frac{q^{\nabla\nabla\nabla}(z_0)}{7!} + b_{-4} \frac{q^{\nabla\nabla}(z_0)}{6!} + b_{-3} \frac{q^{\nabla}(z_0)}{5!} = \frac{p''(z_0)}{2!}$$

$$b_{-3} \frac{q^{\nabla}(z_0)}{5!} = \frac{p''(z_0)}{2!} - b_{-5} \frac{q^{\nabla\nabla\nabla}(z_0)}{7!} - b_{-4} \frac{q^{\nabla\nabla}(z_0)}{6!}$$

$$= \frac{p''(z_0)}{2!} - \frac{5!}{q^{\nabla}(z_0)} \times \frac{q^{\nabla\nabla\nabla}(z_0) p(z_0)}{7!} - \frac{5!}{q^{\nabla}(z_0)} \times \frac{q^{\nabla\nabla}(z_0) p'(z_0)}{6!} +$$

$$\left[\frac{5!}{q^{\nabla}(z_0)} \times \frac{q^{\nabla\nabla}(z_0)}{6!} \right]^2 p(z_0)$$

$$b_{-3} = \frac{5!}{q^{\nabla}(z_0)} \times \frac{p''(z_0)}{2!} - \left[\frac{5!}{q^{\nabla}(z_0)} \right] \frac{q^{\nabla\nabla\nabla}(z_0) p(z_0)}{7!} -$$

$$\left[\frac{5!}{q^{\nabla}(z_0)} \right] \frac{q^{\nabla\nabla}(z_0) p'(z_0)}{6!} + \left[\frac{5!}{q^{\nabla}(z_0)} \right] \left[\frac{q^{\nabla\nabla}(z_0)}{6!} \right]^2 p(z_0)$$

$$b_{-5} \frac{q^{\nabla\nabla\nabla\nabla}(z_0)}{8!} + b_{-4} \frac{q^{\nabla\nabla\nabla}(z_0)}{7!} + b_{-3} \frac{q^{\nabla\nabla}(z_0)}{6!} + b_{-2} \frac{q^{\nabla}(z_0)}{5!} = \frac{p'''(z_0)}{3!}$$

$$b_{-2} \frac{q^{\nabla}(z_0)}{5!} = \frac{p'''(z_0)}{3!} - b_{-5} \frac{q^{\nabla\nabla\nabla\nabla}(z_0)}{8!} - b_{-4} \frac{q^{\nabla\nabla\nabla}(z_0)}{7!} - b_{-3} \frac{q^{\nabla\nabla}(z_0)}{6!}$$

$$= \frac{p'''(z_0)}{3!} - \frac{5!}{q^{\nabla}(z_0)} \times \frac{q^{\nabla\nabla\nabla\nabla}(z_0) p(z_0)}{8!} - \frac{5!}{q^{\nabla}(z_0)} \times \frac{q^{\nabla\nabla\nabla}(z_0) p'(z_0)}{7!}$$

$$\begin{aligned}
 & + \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \frac{q^{\nabla}(z_0) q^{\nabla}(z_0) p(z_0)}{6! 7!} - \frac{5! q^{\nabla}(z_0) p''(z_0)}{q^{\nabla}(z_0) 6! 2!} \\
 & + \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \frac{q^{\nabla}(z_0) q^{\nabla}(z_0) p(z_0)}{6! 7!} + \left[\frac{5! q^{\nabla}(z_0)}{q^{\nabla}(z_0) 6!} \right] \times p'(z_0) - \\
 & \left[\frac{5! q^{\nabla}(z_0)}{q^{\nabla}(z_0) 6!} \right] \times p(z_0) \\
 b_{-2} = & \frac{5! p'''(z_0)}{q^{\nabla}(z_0) 3!} - \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \frac{q^{\nabla}(z_0) p(z_0)}{8!} - \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \frac{q^{\nabla}(z_0) p'(z_0)}{7!} \\
 & + \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \frac{q^{\nabla}(z_0) q^{\nabla}(z_0) p(z_0)}{6! 7!} - \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \frac{q^{\nabla}(z_0) p''(z_0)}{6! 2!} + \\
 & \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \frac{q^{\nabla}(z_0) q^{\nabla}(z_0) p(z_0)}{6! 7!} + \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \left[\frac{q^{\nabla}(z_0)}{6!} \right]^2 \times p'(z_0) - \\
 & \left[\frac{5!}{q^{\nabla}(z_0)} \right] \times \left[\frac{q^{\nabla}(z_0)}{6!} \right]^3 \times p(z_0) \\
 b_{-5} \frac{q^{\nabla}(z_0)}{9!} + b_{-4} \frac{q^{\nabla}(z_0)}{8!} + b_{-3} \frac{q^{\nabla}(z_0)}{7!} + b_{-2} \frac{q^{\nabla}(z_0)}{6!} + b_{-1} \frac{q^{\nabla}(z_0)}{5!} \\
 & = \frac{p^{\nabla}(z_0)}{4!} \\
 b_{-1} \frac{q^{\nabla}(z_0)}{5!} = & \frac{p^{\nabla}(z_0)}{4!} - b_{-5} \frac{q^{\nabla}(z_0)}{9!} - b_{-4} \frac{q^{\nabla}(z_0)}{8!} - b_{-3} \frac{q^{\nabla}(z_0)}{7!} - \\
 & b_{-2} \frac{q^{\nabla}(z_0)}{6!}
 \end{aligned}$$

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$$\begin{aligned}
 &= \frac{p^{IV}(z_0)}{4!} \frac{5!}{q^{\underline{V}}(z_0)} \times \frac{q^{IX}(z_0)p(z_0)}{9!} - \frac{5!}{q^{\underline{V}}(z_0)} \frac{q^{VIII}(z_0)p'(z_0)}{8!} + \\
 &\left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^2 \frac{q^{\underline{VI}}(z_0)q^{\underline{VIII}}(z_0)p(z_0)}{6!8!} - \frac{5!}{q^{\underline{V}}(z_0)} \frac{q^{\underline{VII}}(z_0)p''(z_0)}{7!2!} + \\
 &\left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^2 \frac{q^{\underline{VIII}}(z_0)}{7!} p(z_0) + \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^2 \frac{q^{\underline{VI}}(z_0)q^{\underline{VII}}(z_0)p'(z_0)}{6!7!} - \\
 &\left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^3 \frac{q^{\underline{VI}}(z_0)q^{\underline{VII}}(z_0)p(z_0)}{6!7!} - \frac{5!}{q^{\underline{V}}(z_0)} \frac{q^{\underline{VI}}(z_0)p'''(z_0)}{6!3!} + \\
 &\left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^2 \frac{q^{\underline{VI}}(z_0)q^{\underline{VIII}}(z_0)p(z_0)}{6!8!} + \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^2 \frac{q^{\underline{VI}}(z_0)q^{\underline{VII}}(z_0)p'(z_0)}{6!7!} \\
 &\left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^3 \frac{q^{\underline{VI}}(z_0)q^{\underline{VII}}(z_0)p(z_0)}{6!7!} + \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^2 \frac{q^{\underline{VII}}(z_0)}{6!} p''(z_0) \\
 &- \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^3 \frac{q^{\underline{VI}}(z_0)q^{\underline{VII}}(z_0)p(z_0)}{6!7!} - \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^3 \frac{q^{\underline{VI}}(z_0)}{6!} p'(z_0) \\
 &+ \left[\frac{5!}{q^{\underline{V}}(z_0)} \times \frac{q^{\underline{VI}}(z_0)}{6!} \right]^4 p(z_0) \\
 &= \frac{q^{\underline{V}}(z_0)}{5!} \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^4 \left\{ \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^4 \frac{p^{IV}(z_0)}{4!} - \left[\frac{5!}{q^{\underline{V}}(z_0)} \right]^3 \frac{q^{\underline{IV}}(z_0)p(z_0)}{9!} \right\}
 \end{aligned}$$

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$$\begin{aligned}
 & \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 \frac{q^{\text{VIII}}(z_0) p'(z_0)}{8!} + \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VI}}(z_0) q^{\text{VIII}}(z_0) p(z_0)}{6! 8!} - \\
 & \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 \frac{q^{\text{VII}}(z_0) p''(z_0)}{7! 2!} + \left[\frac{q^{\text{V}}(z_0) q^{\text{VII}}(z_0)}{5! 7!} \right]^2 p(z_0) + \\
 & \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p'(z_0)}{6! 7!} - \frac{q^{\text{V}}(z_0)}{5!} \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \frac{q^{\text{VII}}(z_0) p(z_0)}{7!} \\
 & - \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VI}}(z_0) p'''(z_0)}{6! 3!} + \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p(z_0)}{6! 8!} + \\
 & \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p'(z_0)}{6! 7!} - \frac{q^{\text{V}}(z_0)}{5!} \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \frac{q^{\text{VII}}(z_0) p(z_0)}{7!} \\
 & + \left[\frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0)}{5! 6!} \right]^2 \frac{p''(z_0)}{2!} - \frac{q^{\text{V}}(z_0)}{5!} \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \frac{q^{\text{VII}}(z_0) p(z_0)}{7!} \\
 & - \left. \frac{q^{\text{V}}(z_0)}{5!} \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] p'(z_0) + \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^4 p(z_0) \right\} \\
 b) \quad & \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^5 - \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^4 \frac{p^{\text{IV}}(z_0)}{4!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 \frac{q^{\text{VI}}(z_0) p'''(z_0)}{6! 3!} + \\
 & \left[\frac{q^{\text{V}}(z_0) q^{\text{VII}}(z_0)}{5! 6!} \right]^2 \frac{p''(z_0)}{2!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 \frac{q^{\text{VII}}(z_0) p''(z_0)}{7! 2!} - \\
 & \frac{q^{\text{V}}(z_0)}{5!} \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^3 p'(z_0) + \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p'(z_0)}{6! 7!} +
 \end{aligned}$$

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$$\begin{aligned}
 & \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p'(z_0)}{6! \cdot 7!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{VIII}}(z_0) p'(z_0)}{8!} + \\
 & \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \times p(z_0) - \frac{q^{\text{V}}(z_0)}{5!} \times \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \times \frac{q^{\text{VII}}(z_0) p(z_0)}{7!} - \\
 & \frac{q^{\text{V}}(z_0)}{5!} \times \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \times \frac{q^{\text{VII}}(z_0) p(z_0)}{7!} + \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{VI}}(z_0) q^{\text{VIII}}(z_0) p(z_0)}{6! \cdot 8!} \\
 & - \frac{q^{\text{V}}(z_0)}{5!} \times \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \times \frac{q^{\text{VII}}(z_0) p(z_0)}{7!} + \left[\frac{q^{\text{V}}(z_0) q^{\text{VII}}(z_0)}{5! \cdot 7!} \right] \times p(z_0) + \\
 & \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{VI}}(z_0) q^{\text{VIII}}(z_0) p(z_0)}{6! \cdot 8!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{IX}}(z_0) p(z_0)}{9!} \\
 = & \frac{q^{\text{V}}(z_0)}{5!} \left\{ \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{p^{\text{IV}}(z_0)}{4!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{VI}}(z_0) p^{\text{III}}(z_0)}{6! \cdot 3!} + \right. \\
 & \left. \frac{q^{\text{V}}(z_0)}{5!} \times \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \times \frac{p^{\text{II}}(z_0)}{2!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{VII}}(z_0) p^{\text{II}}(z_0)}{7! \cdot 2!} - \right. \\
 & \left. \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \times p'(z_0) + \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p'(z_0)}{5! \cdot 6! \cdot 7!} + \right. \\
 & \left. \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p'(z_0)}{5! \cdot 6! \cdot 7!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right] \times \frac{q^{\text{VIII}}(z_0) p'(z_0)}{8!} \right\} \\
 & \frac{q^{\text{VI}}(z_0)}{6!} \left\{ - \left[\frac{q^{\text{VI}}(z_0)}{6!} \right] \times p(z_0) + \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) q^{\text{VII}}(z_0) p(z_0)}{5! \cdot 6! \cdot 7!} + \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left. \frac{q^{\text{V}}(z_0)q^{\text{VI}}(z_0)q^{\text{VII}}(z_0)p(z_0)}{5!6!7!} - \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VII}}(z_0)p(z_0)}{8!} \right\} + \\
 & \frac{q^{\text{VII}}(z_0)}{7!} \left\{ \frac{q^{\text{V}}(z_0) \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^2 p(z_0)}{5!} + \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{VII}}(z_0)p(z_0)}{7!} \right\} \\
 & - \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \frac{q^{\text{V}}(z_0) \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^2 p(z_0)}{5!} \right\} - \frac{q^{\text{IX}}(z_0) \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 p(z_0)}{9!} \\
 & = \frac{q^{\text{V}}(z_0)}{5!} \left\{ \frac{q^{\text{V}}(z_0) \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 \frac{q^{\text{IV}}(z_0)p(z_0)}{4!} - \frac{q^{\text{V}}(z_0)q^{\text{VI}}(z_0)p^{\text{III}}(z_0)}{5!6!3!} \right. \\
 & \left. + \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^2 \frac{p''(z_0)}{2!} - \frac{q^{\text{V}}(z_0)q^{\text{VII}}(z_0)p''(z_0)}{5!7!2!} \right\} - \\
 & \frac{q^{\text{VI}}(z_0)}{6!} \left\{ \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^2 p'(z_0) - \frac{q^{\text{V}}(z_0)q^{\text{VII}}(z_0)p'(z_0)}{5!7!} \right\} + \\
 & \frac{q^{\text{VII}}(z_0)q^{\text{V}}(z_0)q^{\text{VI}}(z_0)p'(z_0)}{7!5!6!} - \frac{q^{\text{VIII}}(z_0) \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 p'(z_0)}{8!} \left. \right\} - \\
 & \frac{q^{\text{VI}}(z_0)}{6!} \left\{ \frac{q^{\text{VI}}(z_0)}{6!} \left\{ \left[\frac{q^{\text{VII}}(z_0)}{6!} \right]^2 p(z_0) - \frac{q^{\text{V}}(z_0)q^{\text{VII}}(z_0)p(z_0)}{5!7!} \right\} \right. \\
 & \left. + \frac{q^{\text{VII}}(z_0)q^{\text{V}}(z_0)q^{\text{VI}}(z_0)p(z_0)}{7!5!6!} - \frac{q^{\text{VIII}}(z_0) \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 p(z_0)}{8!} \right\} + \\
 & \frac{q^{\text{VII}}(z_0)}{7!} \left\{ \frac{q^{\text{V}}(z_0) \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^2 p(z_0)}{5!} - \frac{q^{\text{V}}(z_0)q^{\text{VII}}(z_0)p(z_0)}{5!7!} \right\} -
 \end{aligned}$$

$$\begin{aligned}
 & \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ - \frac{[q^{\text{V}}(z_0)]^2}{5!} \frac{q^{\text{VI}}(z_0) p(z_0)}{6!} \right\} - \frac{q^{\text{IX}}(z_0)}{9!} \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^3 p(z_0) \\
 & = \frac{q^{\text{V}}(z_0)}{5!} \left[\frac{q^{\text{V}}(z_0)}{5!} \left\{ \frac{[q^{\text{V}}(z_0)]^2}{5!} \frac{p^{\text{IV}}(z_0)}{4!} - \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) p^{\text{III}}(z_0)}{5! \cdot 6! \cdot 3!} + \right. \right. \\
 & \quad \left. \left. \frac{[q^{\text{VI}}(z_0)]^2}{6!} \frac{p^{\text{II}}(z_0)}{2!} - \frac{q^{\text{V}}(z_0) q^{\text{VII}}(z_0) p^{\text{II}}(z_0)}{5! \cdot 7! \cdot 2!} \right\} - \right. \\
 & \quad \left. \frac{q^{\text{VI}}(z_0)}{6!} \left\{ 0 + \left[\frac{q^{\text{VI}}(z_0)}{6!} \right]^2 \frac{p'(z_0)}{5!} - \frac{q^{\text{V}}(z_0) q^{\text{VII}}(z_0) p'(z_0)}{5! \cdot 7!} \right\} + \right. \\
 & \quad \frac{q^{\text{VII}}(z_0)}{7!} \left\{ 0 + \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) p'(z_0)}{5! \cdot 6!} + 0 \right\} - \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ 0 + \right. \\
 & \quad \left. \frac{[q^{\text{V}}(z_0)]^2}{5!} \frac{p'(z_0)}{5!} + 0 \right\} - \frac{q^{\text{VI}}(z_0)}{6!} \left\{ 0 \right\} - \frac{q^{\text{VI}}(z_0)}{6!} \left\{ 0 \right\} + \\
 & \quad \left. \frac{[q^{\text{VI}}(z_0)]^2}{6!} \frac{p(z_0)}{5!} - \frac{q^{\text{V}}(z_0) q^{\text{VII}}(z_0) p(z_0)}{5! \cdot 7!} \right\} + \frac{q^{\text{VII}}(z_0)}{7!} \left\{ 0 \right\} \\
 & \quad \left. \frac{q^{\text{V}}(z_0) q^{\text{VI}}(z_0) p(z_0)}{5! \cdot 6!} + 0 \right\} - \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ 0 + \frac{[q^{\text{V}}(z_0)]^2}{5!} \frac{p(z_0)}{5!} + 0 \right\} \\
 & + \frac{q^{\text{VII}}(z_0)}{7!} \left\{ 0 \right\} - \frac{q^{\text{V}}(z_0)}{5!} \left\{ 0 - \frac{[q^{\text{VI}}(z_0)]^2}{6!} \frac{p(z_0)}{5!} - \right. \\
 & \quad \left. \frac{q^{\text{V}}(z_0) q^{\text{VII}}(z_0) p(z_0)}{5! \cdot 7!} \right\} + \frac{q^{\text{VII}}(z_0)}{7!} \left\{ 0 - 0 - 0 \right\}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ 0 - 0 + 0 \right\} - \frac{q^{\text{VIII}}(z_0)}{8!} \left[\left\{ 0 \right\} - \frac{q^{\text{V}}(z_0)}{5!} \left\{ 0 \right\} + \right. \\
 & \left. \frac{q^{\text{V}}(z_0)q^{\text{VI}}(z_0)p(z_0)}{5! \cdot 6!} + 0 \right] + \frac{q^{\text{VI}}(z_0)}{6!} \left\{ 0 - 0 + 0 \right\} - \\
 & \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ 0 - 0 + 0 \right\} + \frac{q^{\text{IX}}(z_0)}{9!} \left\{ 0 \right\} - \frac{q^{\text{V}}(z_0)}{5!} \left\{ 0 \right\} + \left[\frac{q^{\text{V}}(z_0)}{5!} \right]^2 p(z_0) + \\
 & \left. + 0 \right\} + \frac{q^{\text{VI}}(z_0)}{6!} \left\{ 0 - 0 + 0 \right\} - \frac{q^{\text{VII}}(z_0)}{7!} \left\{ 0 - 0 + 0 \right\} \\
 & = \frac{q^{\text{V}}(z_0)}{5!} \left[\frac{q^{\text{V}}(z_0)}{5!} \left\{ \frac{q^{\text{V}}(z_0)}{5!} \right. \right. \left. \left. \begin{array}{c|c} \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{III}}(z_0)}{3!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right| - \right. \\
 & \left. \frac{q^{\text{VI}}(z_0)}{6!} \left\{ \begin{array}{c|c} 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right| + \frac{q^{\text{VII}}(z_0)}{7!} \left\{ \begin{array}{c|c} 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right| \right\} - \\
 & \frac{q^{\text{VI}}(z_0)}{6!} \left\{ \begin{array}{c|c} \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{III}}(z_0)}{3!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right| - \frac{q^{\text{VI}}(z_0)}{6!} \left\{ \begin{array}{c|c} 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right| + \\
 & \frac{q^{\text{VII}}(z_0)}{7!} \left\{ \begin{array}{c|c} 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} + \frac{q^{\text{VII}}(z_0)}{7!} \left\{ \begin{array}{c|c} 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right\} -
 \end{aligned}$$

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$$\begin{aligned}
 & \left. \frac{q^V(z_0)}{5!} \begin{vmatrix} 0 & p'(z_0) \\ q^V(z_0) & p^IV(z_0) \end{vmatrix} + \frac{q^VII(z_0)}{7!} \begin{vmatrix} 0 & p'(z_0) \\ 0 & p''(z_0) \end{vmatrix} \right\} - \\
 & \left. \frac{q^VII(z_0)}{8!} \begin{vmatrix} 0 & p''(z_0) \\ q^V(z_0) & p''''(z_0) \end{vmatrix} - \frac{q^V(z_0)}{5!} \begin{vmatrix} 0 & p'(z_0) \\ q^V(z_0) & p''''(z_0) \end{vmatrix} + \right. \\
 & \frac{q^VI(z_0)}{6!} \begin{vmatrix} 0 & p'(z_0) \\ 0 & p''(z_0) \end{vmatrix} - \frac{q^VI(z_0)}{6!} \begin{vmatrix} q^V(z_0) & p''''(z_0) \\ 5! & q^VI(z_0) & p^IV(z_0) \end{vmatrix} \\
 & \left. \frac{q^VI(z_0)}{6!} \begin{vmatrix} 0 & p''(z_0) \\ q^VI(z_0) & p^IV(z_0) \end{vmatrix} + \frac{q^VII(z_0)}{7!} \begin{vmatrix} 0 & p''(z_0) \\ q^V(z_0) & p''''(z_0) \end{vmatrix} \right\} - \\
 & \left. \frac{q^VI(z_0)}{6!} \begin{vmatrix} q^V(z_0) & p''''(z_0) \\ 5! & q^VI(z_0) & p^IV(z_0) \end{vmatrix} - \frac{q^VI(z_0)}{6!} \begin{vmatrix} 0 & p(z_0) \\ q^VI(z_0) & p^IV(z_0) \end{vmatrix} + \right. \\
 & \left. \frac{q^VII(z_0)}{7!} \begin{vmatrix} 0 & p(z_0) \\ q^V(z_0) & p''''(z_0) \end{vmatrix} + \frac{q^VII(z_0)}{7!} \begin{vmatrix} 0 & p''(z_0) \\ q^VI(z_0) & p^IV(z_0) \end{vmatrix} \right\}
 \end{aligned}$$

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$$\begin{aligned}
 & \left. \begin{aligned} & \frac{q^V(z_0)}{5!} \left\{ \begin{array}{l} 0 \\ q^VI(z_0) \end{array} \right\} \begin{array}{l} p'(z_0) \\ p^IV(z_0) \end{array} \\ & + \frac{q^VII(z_0)}{7!} \left\{ \begin{array}{l} 0 \\ 0 \end{array} \right\} \begin{array}{l} p(z_0) \\ p''(z_0) \end{array} \end{aligned} \right\} \\
 & - \frac{q^VIII(z_0)}{8!} \left\{ \begin{array}{l} 0 \\ q^V(z_0) \end{array} \right\} \begin{array}{l} p''(z_0) \\ p^III(z_0) \end{array} \\
 & + \frac{q^V(z_0)}{5!} \left\{ \begin{array}{l} 0 \\ q^V(z_0) \end{array} \right\} \begin{array}{l} p(z_0) \\ p^III(z_0) \end{array} \quad + \\
 \hline
 & \left. \begin{aligned} & \frac{q^VI(z_0)}{6!} \left\{ \begin{array}{l} 0 \\ 0 \end{array} \right\} \begin{array}{l} p'(z_0) \\ p''(z_0) \end{array} \\ & + \frac{q^VII(z_0)}{7!} \left\{ \begin{array}{l} q^V(z_0) \\ q^VI(z_0) \end{array} \right\} \begin{array}{l} p^III(z_0) \\ p^IV(z_0) \end{array} \end{aligned} \right\} \\
 & \frac{q^VI(z_0)}{6!} \left\{ \begin{array}{l} 0 \\ q^VI(z_0) \end{array} \right\} \begin{array}{l} p'(z_0) \\ p^IV(z_0) \end{array} \\
 & + \frac{q^VII(z_0)}{7!} \left\{ \begin{array}{l} 0 \\ q^V(z_0) \end{array} \right\} \begin{array}{l} p'(z_0) \\ p^III(z_0) \end{array} \quad - \\
 & \frac{q^V(z_0)}{5!} \left\{ \begin{array}{l} q^V(z_0) \\ q^VI(z_0) \end{array} \right\} \begin{array}{l} p^III(z_0) \\ p^IV(z_0) \end{array} \\
 & - \frac{q^VI(z_0)}{6!} \left\{ \begin{array}{l} 0 \\ q^VI(z_0) \end{array} \right\} \begin{array}{l} p(z_0) \\ p^IV(z_0) \end{array} \quad + \\
 & \frac{q^VIII(z_0)}{7!} \left\{ \begin{array}{l} 0 \\ q^V(z_0) \end{array} \right\} \begin{array}{l} p(z_0) \\ p^III(z_0) \end{array} \\
 & + \frac{q^VII(z_0)}{7!} \left\{ \begin{array}{l} 0 \\ q^VI(z_0) \end{array} \right\} \begin{array}{l} p'(z_0) \\ p^IV(z_0) \end{array} \quad +
 \end{aligned}$$

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$$\begin{aligned}
 & - \frac{q^{\text{VI}}(z_0)}{6!} \begin{vmatrix} 0 & p(z_0) \\ q^{\text{VI}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix} + \frac{q^{\text{VII}}(z_0)}{7!} \begin{vmatrix} 0 & p(z_0) \\ 0 & p'(z_0) \end{vmatrix} \\
 & - \frac{q^{\text{VIII}}(z_0)}{8!} \begin{vmatrix} 0 & p'(z_0) \\ q^{\text{V}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix} - \frac{q^{\text{V}}(z_0)}{5!} \begin{vmatrix} 0 & p(z_0) \\ q^{\text{V}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix} \\
 & \frac{q^{\text{VII}}(z_0)}{7!} \begin{vmatrix} 0 & p(z_0) \\ 0 & p'(z_0) \end{vmatrix} + \frac{q^{\text{VIII}}(z_0)}{8!} \begin{vmatrix} 0 & p^{\text{III}}(z_0) \\ q^{\text{VI}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix} \\
 & \frac{q^{\text{V}}(z_0)}{5!} \begin{vmatrix} 0 & p'(z_0) \\ q^{\text{VI}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix} + \frac{q^{\text{VII}}(z_0)}{7!} \begin{vmatrix} 0 & p^{\text{III}}(z_0) \\ 0 & p^{\text{III}}(z_0) \end{vmatrix} \\
 & \frac{q^{\text{V}}(z_0)}{5!} \begin{vmatrix} 0 & p^{\text{III}}(z_0) \\ q^{\text{VI}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix} - \frac{q^{\text{V}}(z_0)}{5!} \begin{vmatrix} 0 & p(z_0) \\ q^{\text{VI}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix} \\
 & + \frac{q^{\text{VII}}(z_0)}{7!} \begin{vmatrix} 0 & p(z_0) \\ 0 & p^{\text{III}}(z_0) \end{vmatrix} + \frac{q^{\text{VI}}(z_0)}{6!} \begin{vmatrix} 0 & p'(z_0) \\ q^{\text{VI}}(z_0) & p^{\text{IV}}(z_0) \end{vmatrix}
 \end{aligned}$$

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$$\begin{aligned}
 & \left. \begin{array}{c} \left. \begin{array}{c} 0 \quad p(z_0) \\ \frac{q^{\text{VII}}(z_0)}{6!} \quad \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right\} + \frac{q^{\text{VII}}(z_0)}{7!} \left. \begin{array}{c} 0 \quad p(z_0) \\ 0 \quad p'(z_0) \end{array} \right\} \\
 \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \begin{array}{c} 0 \quad p'(z_0) \\ 0 \quad \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} - \left. \begin{array}{c} 0 \quad p(z_0) \\ 0 \quad \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} + \\
 \frac{q^{\text{V}}(z_0)}{5!} \left. \begin{array}{c} 0 \quad p(z_0) \\ 0 \quad p'(z_0) \end{array} \right\} - \frac{q^{\text{IV}}(z_0)}{9!} \left. \begin{array}{c} 0 \quad \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{5!} \quad \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} \\
 \frac{q^{\text{V}}(z_0)}{5!} \left. \begin{array}{c} 0 \quad p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} \quad \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} + \frac{q^{\text{VI}}(z_0)}{6!} \left. \begin{array}{c} 0 \quad p'(z_0) \\ 0 \quad \frac{p^{\text{II}}(z_0)}{2!} \end{array} \right\} - \\
 \frac{q^{\text{V}}(z_0)}{5!} \left. \begin{array}{c} 0 \quad \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{V}}(z_0)}{5!} \quad \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} - \frac{q^{\text{V}}(z_0)}{5!} \left. \begin{array}{c} 0 \quad p(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} \quad \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} + \\
 \frac{q^{\text{VI}}(z_0)}{6!} \left. \begin{array}{c} 0 \quad p(z_0) \\ 0 \quad \frac{p^{\text{II}}(z_0)}{2!} \end{array} \right\} + \frac{q^{\text{VI}}(z_0)}{6!} \left. \begin{array}{c} 0 \quad p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} \quad \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} -
 \end{aligned}$$

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$$\begin{aligned}
 & \left. \begin{array}{c} \left. \begin{array}{c} \left. \begin{array}{c} 0 \quad p(z_0) \\ \frac{q^V(z_0)}{5!} \quad \frac{p^{III}(z_0)}{3!} \end{array} \right\} + \frac{q^VI(z_0)}{6!} \quad \left. \begin{array}{c} 0 \quad p(z_0) \\ 0 \quad p'(z_0) \end{array} \right\} \\ \frac{q^VII(z_0)}{7!} \left\{ \begin{array}{c} 0 \quad p'(z_0) \\ 0 \quad \frac{p^{II}(z_0)}{2!} \end{array} \right\} - \left. \begin{array}{c} 0 \quad p(z_0) \\ 0 \quad \frac{p^{II}(z_0)}{2!} \end{array} \right\} \\ \hline + \frac{q^V(z_0)}{5!} \left. \begin{array}{c} 0 \quad p(z_0) \\ 0 \quad p'(z_0) \end{array} \right\} \\ \hline = \frac{q^V(z_0)}{5!} \left\{ \begin{array}{c} \frac{q^V(z_0)}{5!} \quad 0 \quad \frac{p^{II}(z_0)}{2!} \\ \frac{q^VI(z_0)}{6!} \quad \frac{q^V(z_0)}{5!} \quad \frac{p^{III}(z_0)}{3!} \\ \frac{q^VII(z_0)}{7!} \quad \frac{q^VI(z_0)}{6!} \quad \frac{p^IV(z_0)}{4!} \end{array} \right\} \\ \hline \frac{q^VI(z_0)}{6!} \quad \frac{q^VI(z_0)}{6!} \quad \frac{q^V(z_0)}{5!} \quad \frac{p^{III}(z_0)}{3!} + \frac{q^VII(z_0)}{7!} \quad \frac{q^V(z_0)}{5!} \quad 0 \quad \frac{p'(z_0)}{2!} \\ \frac{q^VII(z_0)}{7!} \quad \frac{q^VI(z_0)}{6!} \quad \frac{p^IV(z_0)}{4!} \quad \frac{q^VII(z_0)}{7!} \quad \frac{q^VI(z_0)}{6!} \quad \frac{p^IV(z_0)}{4!}
 \end{array}
 \end{aligned}$$

$\frac{\frac{q^{\text{VIII}}(z_0)}{8!} \frac{q^{\text{V}}(z_0)}{5!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0)}{p'(z_0)}$	}	$\frac{q^{\text{VI}}(z_0)}{6!} \left\{ \begin{array}{l} \frac{q^{\text{V}}(z_0)}{5!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0) \\ \frac{q^{\text{VII}}(z_0)}{7!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{IV}}(z_0) \end{array} \right\} p'(z_0)$
$\frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0)$	+	$\frac{q^{\text{VII}}(z_0)}{7!} \frac{q^{\text{V}}(z_0)}{5!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0)$
$\frac{q^{\text{VIII}}(z_0)}{8!} \frac{q^{\text{V}}(z_0)}{5!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0)$	}	$\frac{q^{\text{VII}}(z_0)}{7!} \left\{ \begin{array}{l} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0) \\ \frac{q^{\text{VII}}(z_0)}{7!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{IV}}(z_0) \end{array} \right\} p'(z_0)$
$\frac{q^{\text{V}}(z_0)}{5!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0)$	+	$\frac{q^{\text{VII}}(z_0)}{7!} \left\{ \begin{array}{l} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{III}}(z_0) \\ \frac{q^{\text{VII}}(z_0)}{7!} \frac{q^{\text{VI}}(z_0)}{6!} \frac{q^{\text{V}}(z_0)}{5!} p^{\text{IV}}(z_0) \end{array} \right\} p'(z_0)$

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$$\begin{array}{c}
 \left. \begin{array}{c}
 \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ 0 & 0 & p'(z_0) \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right\} \\
 \\
 \frac{q^{\text{V}}(z_0)}{5!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ q^{\text{V}}(z_0) & 0 & \frac{p^{\text{III}}(z_0)}{3!} \\ \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right\} \\
 \\
 \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ 0 & 0 & p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & 0 & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} \\
 \\
 \frac{q^{\text{V}}(z_0)}{5!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ q^{\text{V}}(z_0) & 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\}
 \end{array}
 \right\} + \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \begin{array}{ccc} 0 & 0 & p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & 0 & \frac{p^{\text{III}}(z_0)}{3!} \\ \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right\} \\
 \\
 \left. \begin{array}{c}
 \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \begin{array}{ccc} 0 & 0 & p'(z_0) \\ 0 & 0 & p''(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & 0 & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} \\
 \\
 \frac{q^{\text{V}}(z_0)}{5!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ q^{\text{V}}(z_0) & 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} \\
 \\
 \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ 0 & 0 & p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & 0 & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\} \\
 \\
 \frac{q^{\text{V}}(z_0)}{5!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ q^{\text{V}}(z_0) & 0 & \frac{p^{\text{II}}(z_0)}{2!} \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & \frac{p^{\text{III}}(z_0)}{3!} \end{array} \right\}
 \end{array}
 \right\} + \frac{q^{\text{VIII}}(z_0)}{8!} \left\{ \begin{array}{ccc} 0 & 0 & p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & 0 & \frac{p^{\text{III}}(z_0)}{3!} \\ \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{VI}}(z_0)}{6!} & \frac{p^{\text{IV}}(z_0)}{4!} \end{array} \right\}
 \end{array}$$

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$$\begin{aligned}
 & \frac{q^{\text{VII}}(z_0)}{7!} \left\{ \begin{array}{ccc} 0 & 0 & p(z_0) \\ 0 & 0 & p'(z_0) \\ \frac{q^{\text{V}}(z_0)}{5!} & 0 & p''(z_0) \end{array} \right\} \\
 & = \frac{q^{\text{V}}(z_0)}{5!} \left[\begin{array}{cccc} \frac{q^{\text{V}}(z_0)}{5!} & 0 & 0 & p'(z_0) \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & 0 & p''(z_0) \\ \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & p'''(z_0) \\ \frac{q^{\text{VIII}}(z_0)}{8!} & \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{VI}}(z_0)}{6!} & p^{\text{IV}}(z_0) \\ 0 & 0 & 0 & p(z_0) \\ \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & 0 & p''(z_0) \\ \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{VI}}(z_0)}{6!} & \frac{q^{\text{V}}(z_0)}{5!} & p'''(z_0) \\ \frac{q^{\text{VIII}}(z_0)}{8!} & \frac{q^{\text{VII}}(z_0)}{7!} & \frac{q^{\text{VI}}(z_0)}{6!} & p^{\text{IV}}(z_0) \end{array} \right] +
 \end{aligned}$$

$$\frac{q^{\text{VII}}(z_0)}{7!}$$

0	0	0	$p(z_0)$
$\frac{q^{\text{V}}(z_0)}{5!}$	0	0	$p'(z_0)$
$\frac{q^{\text{VII}}(z_0)}{7!}$	$\frac{q^{\text{VI}}(z_0)}{6!}$	$\frac{q^{\text{V}}(z_0)}{5!}$	$\frac{p''(z_0)}{2!}$
$\frac{q^{\text{VIII}}(z_0)}{8!}$	$\frac{q^{\text{VII}}(z_0)}{7!}$	$\frac{q^{\text{VI}}(z_0)}{6!}$	$\frac{p''''(z_0)}{4!}$

$$\frac{q^{\text{VIII}}(z_0)}{8!}$$

0	0	0	$p(z_0)$
$\frac{q^{\text{V}}(z_0)}{5!}$	0	0	$p'(z_0)$
$\frac{q^{\text{VI}}(z_0)}{6!}$	$\frac{q^{\text{V}}(z_0)}{5!}$	0	$\frac{p''(z_0)}{2!}$
$\frac{q^{\text{VIII}}(z_0)}{8!}$	$\frac{q^{\text{VII}}(z_0)}{7!}$	$\frac{q^{\text{VI}}(z_0)}{6!}$	$\frac{p''''(z_0)}{4!}$

$$\frac{q^{\text{IX}}(z_0)}{9!}$$

0	0	0	$p(z_0)$
$\frac{q^{\text{V}}(z_0)}{5!}$	0	0	$p'(z_0)$
$\frac{q^{\text{VI}}(z_0)}{6!}$	$\frac{q^{\text{V}}(z_0)}{5!}$	0	$\frac{p''(z_0)}{2!}$
$\frac{q^{\text{VII}}(z_0)}{7!}$	$\frac{q^{\text{VI}}(z_0)}{6!}$	$\frac{q^{\text{V}}(z_0)}{5!}$	$\frac{p''''(z_0)}{4!}$

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$$b_{-1} \left[\frac{q^V(z_0)}{5!} \right]^5 =$$

$\frac{q^V(z_0)}{5!}$	0	0	0	$p(z_0)$
$\frac{q^{VI}(z_0)}{6!}$	$\frac{q^V(z_0)}{5!}$	0	0	$p'(z_0)$
$\frac{q^{VII}(z_0)}{7!}$	$\frac{q^{VI}(z_0)}{6!}$	$\frac{q^V(z_0)}{5!}$	0	$p''(z_0)$
$\frac{q^{VIII}(z_0)}{8!}$	$\frac{q^{VII}(z_0)}{7!}$	$\frac{q^{VI}(z_0)}{6!}$	$\frac{q^V(z_0)}{5!}$	$p'''(z_0)$
$\frac{q^IX(z_0)}{9!}$	$\frac{q^{VIII}(z_0)}{8!}$	$\frac{q^{VII}(z_0)}{7!}$	$\frac{q^{VI}(z_0)}{6!}$	$p^{IV}(z_0)$

$$\therefore \text{Res}(f(z), z_0) = b_{-1} \left[\frac{5!}{q^V(z_0)} \right]^5$$

$\frac{q^V(z_0)}{5!}$	0	0	0	$p(z_0)$
$\frac{q^{VI}(z_0)}{6!}$	$\frac{q^V(z_0)}{5!}$	0	0	$p'(z_0)$
$\frac{q^{VII}(z_0)}{7!}$	$\frac{q^{VI}(z_0)}{6!}$	$\frac{q^V(z_0)}{5!}$	0	$p''(z_0)$
$\frac{q^{VIII}(z_0)}{8!}$	$\frac{q^{VII}(z_0)}{7!}$	$\frac{q^{VI}(z_0)}{6!}$	$\frac{q^V(z_0)}{5!}$	$p'''(z_0)$
$\frac{q^IX(z_0)}{9!}$	$\frac{q^{VIII}(z_0)}{8!}$	$\frac{q^{VII}(z_0)}{7!}$	$\frac{q^{VI}(z_0)}{6!}$	$p^{IV}(z_0)$

ทฤษฎีที่ 7 ให้ $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ โดยที่ $p(z)$ และ $q(z)$ สอดคล้องกับเงื่อนไข $q(z_0) = q'(z_0) = q''(z_0) = q'''(z_0) = \dots = q^{(n-1)}(z_0) = 0$ แต่ $q^{(n)}(z_0) \neq 0$ และ $p(z_0) \neq 0$ แล้วฟังก์ชัน $f(z)$ จะมีโพลอันดับที่ n และค่าเรซิดิวของ $f(z)$ จะมีค่าดังนี้

$$\text{Res}(f(z), z_0) = \left[\frac{p(z)}{q^{(n)}(z_0)} \right]_{z=z_0} \times n!$$

$\frac{q^{(n)}(z_0)}{n!}$	0	0	0	$p(z_0)$
$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{q^{(n)}(z_0)}{n!}$	0	0	$p'(z_0)$
$\frac{q^{(n+2)}(z_0)}{(n+2)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{q^{(n)}(z_0)}{n!}$	0	$\frac{p''(z_0)}{2!}$
$\frac{q^{(n+3)}(z_0)}{(n+3)!}$	$\frac{q^{(n+2)}(z_0)}{(n+2)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	0	$\frac{p'''(z_0)}{3!}$
.....
$\frac{q^{(2n-2)}(z_0)}{(2n-2)!}$	$\frac{q^{(2n-3)}(z_0)}{(2n-3)!}$	$\frac{q^{(2n-4)}(z_0)}{(2n-4)!}$	$\frac{q^{(n)}(z_0)}{n!}$	$\frac{p^{(n-2)}(z_0)}{(n-2)!}$
$\frac{q^{(2n-1)}(z_0)}{(2n-1)!}$	$\frac{q^{(2n-2)}(z_0)}{(2n-2)!}$	$\frac{q^{(2n-3)}(z_0)}{(2n-3)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{p^{(n-1)}(z_0)}{(n-1)!}$
				 (5.7)

พิสูจน์ กำหนดให้ $f(z) = \frac{p(z)}{q(z)}$ โดยที่ $p(z)$ และ $q(z)$ ต่างก็เป็นฟังก์ชันวิเคราะห์ ที่จุด $z = z_0$ ซึ่ง $p(z_0) \neq 0$ และ $q(z_0) = q'(z_0) = q''(z_0) = q'''(z_0) = \dots = q^{(n-1)}(z_0) = 0$ แต่ $q^{(n)}(z_0) \neq 0$

แสดงว่า $q(z)$ มีซีโรอันดับที่ m ที่จุด $z = z_0$
 ดังนั้นจากทฤษฎีที่ 4 บทที่ 4 จะได้ว่า $f(z)$ มีโพลอันดับที่ n

กระจายฟังก์ชันวิเคราะห์ $p(z)$ และ $q(z)$ โดยใช้ออนุกรมของเทย์เลอร์

$$f(z) = \frac{p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)(z-z_0)^2}{2!} + \frac{p'''(z_0)(z-z_0)^3}{3!} + \dots}{q^{(n)}(z_0)(z-z_0)^n + \frac{q^{(n+1)}(z_0)(z-z_0)^{(n+1)}}{(n+1)!} + \frac{q^{(n+2)}(z_0)(z-z_0)^{(n+2)}}{(n+2)!} + \dots}$$

$$f(z) = \frac{b_{-n}}{(z-z_0)^n} + \frac{b_{-n+1}}{(z-z_0)^{(n-1)}} + \frac{b_{-n+2}}{(z-z_0)^{(n-2)}} + \frac{b_{-n+3}}{(z-z_0)^{(n-3)}} + \dots$$

$$\dots + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$$

จาก $f(z) = \frac{p(z)}{q(z)}$

$$p(z) = q(z)f(z)$$

$$p(z_0) + p'(z_0)(z-z_0) + \frac{p''(z_0)(z-z_0)^2}{2!} + \frac{p'''(z_0)(z-z_0)^3}{3!} + \dots$$

$$\dots + \frac{p^{(n-1)}(z_0)(z-z_0)^{(n-1)}}{(n-1)!} + \frac{p^{(m)}(z_0)(z-z_0)^m}{n!} + \dots$$

$$= \left[\frac{q^{(n)}(z_0)(z-z_0)^n}{n!} + \frac{q^{(n+1)}(z_0)(z-z_0)^{(n+1)}}{(n+1)!} + \frac{q^{(n+2)}(z_0)(z-z_0)^{(n+2)}}{(n+2)!} \right.$$

$$\left. + \dots + \frac{q^{(n-1)}(z_0)(z-z_0)^{(n-1)}}{(n-1)!} + \frac{q^{(n)}(z_0)(z-z_0)^n}{n!} + \dots \right]$$

$$\left[\frac{b_{-n}}{(z-z_0)^n} + \frac{b_{-n+1}}{(z-z_0)^{(n-1)}} + \frac{b_{-n+2}}{(z-z_0)^{(n-2)}} + \frac{b_{-n+3}}{(z-z_0)^{(n-3)}} + \dots \right. \\ \left. \dots + \frac{b_{-2}}{(z-z_0)^2} + \frac{b_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots \right]$$

$$= \frac{b_{-n} q^{(n)}(z_0)}{n!} + \left[\frac{b_{-n} q^{(n+1)}(z_0)}{(n+1)!} + b_{-n+1} \frac{q^{(n)}(z_0)}{n!} \right] (z-z_0) +$$

$$\left[\frac{b_{-n} q^{(n+2)}(z_0)}{(n+2)!} + b_{-n+1} \frac{q^{(n+1)}(z_0)}{(n+1)!} + b_{-n+2} \frac{q^{(n)}(z_0)}{n!} \right] (z-z_0)^2$$

$$\left[\frac{b_{-n} q^{(n+3)}(z_0)}{(n+3)!} + b_{-n+1} \frac{q^{(n+2)}(z_0)}{(n+2)!} + b_{-n+2} \frac{q^{(n+1)}(z_0)}{(n+1)!} + \right.$$

$$\left. \frac{b_{-n+3} q^{(n)}(z_0)}{n!} \right] (z-z_0)^3 + \dots + \left[\frac{b_{-n} q^{(2n-1)}(z_0)}{(2n-1)!} + \right.$$

$$\left. \frac{b_{-n+1} q^{(2n-2)}(z_0)}{(2n-2)!} + \frac{b_{-n+2} q^{(2n-3)}(z_0)}{(2n-3)!} + \dots \right]$$

$$\left. \frac{b_{-2} q^{(n-1)}(z_0)}{(n-1)!} + \frac{b_{-1} q^{(n)}(z_0)}{n!} \right] (z-z_0)^{(n-1)} + \dots$$

โดยการเทียบสัมประสิทธิ์จะได้สมการทั้งหมด n สมการดังนี้

$$\frac{b_{-n} q^{(n)}(z_0)}{n!} = p(z_0)$$

$$b_{-n} = \frac{n!}{q^{(n)}(z_0)} p(z_0) \dots (5.7.1)$$

$$\frac{b_{-n} q^{(n+1)}(z_0)}{(n+1)!} + \frac{b_{-n+1} q^{(n)}(z_0)}{n!} = p'(z_0)$$

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$$b_{-n+1} \frac{q^{(n)}(z_0)}{n!} = p'(z_0) - b_{-n} \frac{q^{(n+1)}(z_0)}{(n+1)!} \dots \dots (5.7.2)$$

$$b_{-n} \frac{q^{(n+2)}(z_0)}{(n+2)!} + b_{-n+1} \frac{q^{(n+1)}(z_0)}{(n+1)!} + b_{-n+2} \frac{q^{(n)}(z_0)}{n!} = \frac{p''(z_0)}{2!}$$

$$b_{-n+2} \frac{q^{(n)}(z_0)}{n!} = \frac{p''(z_0)}{2!} - b_{-n} \frac{q^{(n+2)}(z_0)}{(n+2)!} - b_{-n+1} \frac{q^{(n+1)}(z_0)}{(n+1)!} \dots \dots (5.7.3)$$

$$b_{-n} \frac{q^{(n+3)}(z_0)}{(n+3)!} + b_{-n+1} \frac{q^{(n+2)}(z_0)}{(n+2)!} + b_{-n+2} \frac{q^{(n+1)}(z_0)}{(n+1)!} + b_{-n+3} \frac{q^{(n)}(z_0)}{n!} = \frac{p'''(z_0)}{3!} \dots \dots$$

$$b_{-n+3} \frac{q^{(n)}(z_0)}{n!} = \frac{p'''(z_0)}{3!} - b_{-n} \frac{q^{(n+3)}(z_0)}{(n+3)!} - b_{-n+1} \frac{q^{(n+2)}(z_0)}{(n+2)!} - b_{-n+2} \frac{q^{(n+1)}(z_0)}{(n+1)!} \dots \dots (5.7.4)$$

.....

$$b_{-n} \frac{q^{(2n-2)}(z_0)}{(2n-2)!} + b_{-n+1} \frac{q^{(2n-3)}(z_0)}{(2n-3)!} + b_{-n+2} \frac{q^{(2n-4)}(z_0)}{(2n-4)!} +$$

$$b_{-n+3} \frac{q^{(2n-5)}(z_0)}{(2n-5)!} + \dots \dots + b_{-4} \frac{q^{(n+2)}(z_0)}{(n+2)!} + b_{-3} \frac{q^{(n+1)}(z_0)}{(n+1)!} +$$

$$b_{-2} \frac{q^{(n)}(z_0)}{n!} = \frac{p^{(n-2)}(z_0)}{(n-2)!}$$

$$\begin{aligned}
 b_{-2} \frac{q^{(n)}(z_0)}{n!} &= \frac{p^{(n-2)}(z_0)}{(n-2)!} - b_{-n} \frac{q^{(2n-2)}(z_0)}{(2n-2)!} - b_{-n+1} \frac{q^{(2n-3)}(z_0)}{(2n-3)!} - \\
 & b_{-n+2} \frac{q^{(2n-4)}(z_0)}{(2n-4)!} - b_{-n+3} \frac{q^{(2n-5)}(z_0)}{(2n-5)!} - \dots \\
 & - b_{-4} \frac{q^{(n+2)}(z_0)}{(n+2)!} - b_{-3} \frac{q^{(n+1)}(z_0)}{(n+1)!} \dots \dots (5.7.(n-1))
 \end{aligned}$$

$$\begin{aligned}
 & b_{-n} \frac{q^{(2n-1)}(z_0)}{(2n-1)!} + b_{-n+1} \frac{q^{(2n-2)}(z_0)}{(2n-2)!} + b_{-n+2} \frac{q^{(2n-3)}(z_0)}{(2n-3)!} + \\
 & b_{-n+3} \frac{q^{(2n-4)}(z_0)}{(2n-4)!} + \dots \dots + b_{-3} \frac{q^{(n+2)}(z_0)}{(n+2)!} +
 \end{aligned}$$

$$b_{-2} \frac{q^{(n+1)}(z_0)}{(n+1)!} + b_{-1} \frac{q^{(n)}(z_0)}{n!} = \frac{p^{(n-1)}(z_0)}{(n-1)!}$$

$$\begin{aligned}
 b_{-1} \frac{q^{(n)}(z_0)}{n!} &= \frac{p^{(n-1)}(z_0)}{(n-1)!} - b_{-n} \frac{q^{(2n-1)}(z_0)}{(2n-1)!} - b_{-n+1} \frac{q^{(2n-2)}(z_0)}{(2n-2)!} - \\
 & b_{-n+2} \frac{q^{(2n-3)}(z_0)}{(2n-3)!} - \dots \dots
 \end{aligned}$$

$$b_{-3} \frac{q^{(n+2)}(z_0)}{(n+2)!} - b_{-2} \frac{q^{(n+1)}(z_0)}{(n+1)!} \dots \dots (5.7.n)$$

จากสมการที่ 5.7.n ถ้าแทนค่า $b_{-n}, b_{-n+1}, b_{-n+2}, b_{-n+3}, \dots$
 $\dots, b_{-4}, b_{-3}, b_{-2}$ จากสมการที่ 5.7.1 ถึงสมการที่ 5.7.(n-1)
 ลงไปในสมการที่ 5.7.n ซึ่งสามารถทำได้
 ในทำนองเดียวกันกับทฤษฎีที่ 2 ถึงทฤษฎีที่ 6 แล้วจะได้ว่า

$$b_{-1} \frac{q^{(n)}(z_0)}{n!} = \left[\frac{n!}{q^{(n)}(z_0)} \right]^{(n-1)} x$$

$\frac{q^{(n)}(z_0)}{n!}$	0	0	0	$p(z_0)$
$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{q^{(n)}(z_0)}{n!}$	0	0	$p'(z_0)$
$\frac{q^{(n+2)}(z_0)}{(n+2)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{q^{(n)}(z_0)}{n!}$	0	$p''(z_0)$
$\frac{q^{(n+3)}(z_0)}{(n+3)!}$	$\frac{q^{(n+2)}(z_0)}{(n+2)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	0	$p'''(z_0)$
⋮	⋮	⋮	⋮	⋮
$\frac{q^{(2n-2)}(z_0)}{(2n-2)!}$	$\frac{q^{(2n-3)}(z_0)}{(2n-3)!}$	$\frac{q^{(2n-4)}(z_0)}{(2n-4)!}$	$\frac{q^{(n)}(z_0)}{n!}$	$p^{(n-2)}(z_0)$
$\frac{q^{(2n-1)}(z_0)}{(2n-1)!}$	$\frac{q^{(2n-2)}(z_0)}{(2n-2)!}$	$\frac{q^{(2n-3)}(z_0)}{(2n-3)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$p^{(n-1)}(z_0)$

$$\text{Res. } (f(z), z_0) = b_{-1} = \left[\frac{n!}{q^{(n)}(z_0)} \right] \times$$

$\frac{q^{(n)}(z_0)}{n!}$	0	0	0	$p(z_0)$
$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{q^{(n)}(z_0)}{n!}$	0	0	$p'(z_0)$
$\frac{q^{(n+2)}(z_0)}{(n+2)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{q^{(n)}(z_0)}{n!}$	0	$p''(z_0)$
$\frac{q^{(n+3)}(z_0)}{(n+3)!}$	$\frac{q^{(n+2)}(z_0)}{(n+2)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	0	$p'''(z_0)$
⋮	⋮	⋮	⋮	⋮	⋮
$\frac{q^{(2n-2)}(z_0)}{(2n-2)!}$	$\frac{q^{(2n-3)}(z_0)}{(2n-3)!}$	$\frac{q^{(2n-4)}(z_0)}{(2n-4)!}$	$\frac{q^{(n)}(z_0)}{n!}$	$\frac{p^{(n-2)}(z_0)}{(n-2)!}$
$\frac{q^{(2n-1)}(z_0)}{(2n-1)!}$	$\frac{q^{(2n-2)}(z_0)}{(2n-2)!}$	$\frac{q^{(2n-3)}(z_0)}{(2n-3)!}$	$\frac{q^{(n+1)}(z_0)}{(n+1)!}$	$\frac{p^{(n-1)}(z_0)}{(n-1)!}$