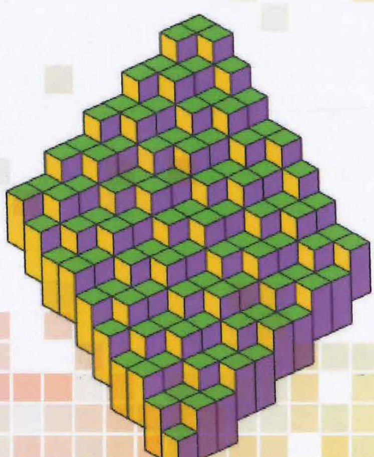
Christer Oscar Kiselman

Elements of
Digital Geometry, Mathematical Morphology, and Discrete Optimization



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Elements of

Digital Geometry, Mathematical Morphology, and Discrete Optimization

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Uppsala University, Sweden

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Contents

Preface v

Acknowledgments ix

List of Figures xxi

List of Tables xxiii

1. Introduction 1

1.1 Why digital geometry? 1
1.2 Why mathematical morphology? 11
1.3 How is image processing at all possible? 13
1.4 Why discrete optimization? 16
1.5 Notes 17
1.6 Exercises 19

2. Sets, mappings, and order relations 21

2.1 Introduction 21
2.2 Sets 21
2.3 Notation for sets of numbers 22
2.4 Counting with infinities 22
2.5 The floor and ceiling functions 23
2.6 Relations and mappings 24
2.7 Preorders 25
2.8 Mappings between preordered sets 27
2.9 Epigraphs and hypographs 29
2.10 Calculating with sets and functions 30
2.11 Cleistomorphisms and anoiktomorphisms 31

2.12 How to measure distribution of sizes? 34
2.13 Notes 34
2.14 Exercises 36

3. Morphological operations: Set-theoretical duality 39

3.1 Groups and semigroups 39
3.2 Dilations and erosions: Set-theoretical duality 41
3.3 Convolution 46
3.4 Infimal convolution 47
3.5 Approximation using infimal convolution 50
3.6 Iterated infimal convolutions 50
3.7 A more general infimal convolution 51
3.8 Restricted Minkowski addition 52
3.9 Combining dilations and erosions 53
3.10 Commuting with translations 55
3.11 Matheron's structural theorems 56
3.12 Notes 61
3.13 Exercises 63

4. Complete lattices 67

4.1 Introduction 67
4.2 Definitions and examples 69
4.3 Moore families and dual Moore families 73
4.4 Dilations and erosions in complete lattices 77
4.5 Notes 78
4.6 Exercises 79

5. Inverses and quotients of mappings 81

5.1 Introduction 81
5.2 Defining inverses of mappings 82
5.3 First properties of inverses 83
5.4 Left inverses 86
5.5 Right inverses 87
5.6 Inverses of inverses 90
5.7 Special cases of inverses 91
5.8 Examples of upper and lower inverses 96
5.9 Quotients of mappings 105
5.10 Pullbacks and pushforwards 108

5.11 Notes 112
5.12 Exercises 113

6. Structure theorems for mappings 115

6.1 Set-theoretical representation of dilations, erosions, cleistomorphisms, and anoiktomorphisms 115
6.2 Anoiktomorphisms as quotients 116
6.3 Increasing mappings as suprema of elementary erosions . . 117
6.4 Anoiktomorphisms as suprema of elementary anoiktomorphisms 118
6.5 Strong ethmomorphisms 119
6.6 Notes 123
6.7 Exercises 123

7. Digitization 125

7.1 Introduction 125
7.2 Serra's four principles on gathering information 128
7.3 Distances and metric spaces 129
7.4 Discrete sets and uniformly discrete sets 131
7.5 To discretize a function 132
7.6 Discretization by balayage 132
7.7 Voronoi cells 132
7.8 Difficulties in logic 135
7.9 Comparing differences and derivatives 136
7.10 Notes 139
7.11 Exercises 140

8. Digital straightness and digital convexity 141

8.1 Introduction 141
8.2 Digital lines and Rosenfeld's chord property 144
8.3 Difference operators 149
8.4 Real-valued functions on the integers 151
8.5 Functions taking real values on $\mathbf{Z}^n$ 153
8.6 Characterization of straightness: The chord property . . . 154
8.7 Characterization of straightness: Balanced words 156
8.8 Hyperplanes in the sense of Reveillès 158
8.9 Refined digital hyperplanes 158
8.10 Extending rectilinear segments 160

8.11 Notes 163
8.12 Exercises 165

9. Convexity in vector spaces 167

9.1 Introduction 167
9.2 Defining convex sets 167
9.3 Properties of convex sets 168
9.4 The convex hull 169
9.5 The Hahn–Banach theorem 170
9.6 Supporting hyperplanes 172
9.7 Carathéodory's theorem 173
9.8 Approximation of the convex hull 173
9.9 Defining convex functions 175
9.10 Properties of convex functions 176
9.11 Strict and strong convexity 177
9.12 Functions taking the value minus infinity 178
9.13 The convex envelope 178
9.14 Approximation of the convex envelope 180
9.15 Separating hyperplanes 184
9.16 Normed spaces 186
9.17 Duality in convex analysis 188
9.18 Duality of infimal convolution and addition 194
9.19 Comparing two convolution operations 194
9.20 Coppel's axioms 196
9.21 Three fundamental properties 196
9.22 Notes 199
9.23 Exercises 201

10. Discrete convexity 203

10.1 Introduction 203
10.2 Restrictions and extensions of functions 203
10.3 Convexity with respect to a subset of a vector space 204
10.4 The integer neighborhood and the canonical extension 205
10.5 Properties of the canonical extension 207
10.6 Integral convexity 209
10.7 Lateral convexity: Definitions 209
10.8 Lateral convexity: Morphological aspects 211
10.9 Lateral convexity: Examples 213

10.10 Representations in terms of elementary convex functions . . . 214
10.11 Separating partially discretized sets . . . 217
10.12 Separating completely discretized sets . . . 219
10.13 Possible future studies . . . 220
10.14 Notes . . . 220
10.15 Exercises . . . 222

11. Discrete convexity in two dimensions 225

11.1 Jensen's inequality in the discrete case . . . 225
11.2 Discrete convexity and the chord property . . . 228
11.3 Submodular and separable functions . . . 230
11.4 The piecewise separately affine extension . . . 231
11.5 Rhomboidal convexity . . . 232
11.6 Conditions for rhomboidal convexity . . . 234
11.7 Independence of the conditions for rhomboidal convexity . . . 239
11.8 The maximal set of pairs which defines rhomboidal convexity . . . 241
11.9 $(\mathbf{Z} \times \mathbf{R})$-convexity and separate $(\mathbf{Z} \times \mathbf{R})$-convexity . . . 242
11.10 Smooth rhomboidally convex functions . . . 242
11.11 Extending functions from integer points . . . 243
11.12 Conditions for digital straightness . . . 245
11.13 What is the result when we digitize a Euclidean line? . . . 247
11.14 Discretizations of a function . . . 249
11.15 Extending convex extensible functions . . . 254
11.16 Possible future studies . . . 255
11.17 Notes . . . 255
11.18 Exercise . . . 257

12. Three problems in discrete optimization 259

12.1 Introduction . . . 259
12.2 Three fundamental problems . . . 259
12.3 Solution to Problem 1: Marginal functions . . . 262
12.4 Solution to Problem 2: Local minima . . . 266
12.5 Solution to Problem 3: Separating hyperplanes . . . 269
12.6 The marginal function of a function of integer variables . . . 275

12.7 Convolution and convex extensibility 276
12.8 The set where the infimum is attained 277
12.9 Lateral convexity of marginal functions 280
12.10 Necessity of lateral convexity 284
12.11 Possible future studies 285
12.12 Notes 286
12.13 Exercises 286

13. Duality of convolution operators 287

13.1 Introduction 287
13.2 Properties of convolution 287
13.3 Families of functions under duality 291
13.4 Duality between classes of functions and second-order difference operators 297
13.5 More general convolution operators 298
13.6 Discrete convexity defined by convolution 300
13.7 Possible future studies 302
13.8 Notes 303
13.9 Exercises 304

14. Topology 305

14.1 Introduction 305
14.2 Mappings 305
14.3 Defining topologies 307
14.4 Transporting topologies 309
14.5 Continuous mappings 310
14.6 Connectedness 310
14.7 Quotient spaces 312
14.8 Separation axioms 313
14.9 Continuous mappings vs. increasing mappings 315
14.10 Matheron's hit-or-miss topology 315
14.11 Notes 315
14.12 Exercises 315

15. The Khalimsky topology 317

15.1 Introduction 317
15.2 Smallest-neighborhood spaces 317
15.3 Continuity for the Khalimsky topology 320

15.4 Digitization of straight lines in the Khalimsky plane . . . 324
15.5 Fixed-point theorems 325
15.6 Jordan curve theorems 331
15.7 Notes 342
15.8 Exercises 343

16. Distance transformations 345

16.1 Defining distances 345
16.2 Translation-invariant distances 345
16.3 Distance transforms 347
16.4 Distance transforms and sublevel sets 352
16.5 Finitely generated distances 355
16.6 Comparing distances 360
16.7 The calculus of balls 363
16.8 Distance transforms in normed vector spaces 372
16.9 Notes 375
16.10 Exercises 377

17. Skeletonizing 381

17.1 Introduction 381
17.2 Defining skeletons, medial axes, and nervures 381
17.3 Comparing medial axes, nervures, and skeletons 383
17.4 Existence of skeletons 385
17.5 Properties of skeletons 387
17.6 Notes 389
17.7 Exercises 391

18. Solutions 393

Bibliography 419

Author Index 449

Subject Index 453