COINCIDENCE POINT THEOREMS FOR SOME GENERALIZED CONTRACTIONS IN GENERALIZED METRIC SPACES

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IN MATHEMATICS

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IN MATHEMATICS

GRADUATE SCHOOL, CHIANG MAI UNIVERSITY
MARCH 2019
To
My Family
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Atit Wiriyapongsanon
หัวข้องุณฑีนิพนธ์

ทฤษฎีบทจุดร่วมสำหรับบางการหดตัวที่วางนัยทั่วไปในปริภูมิเมตริกที่วางนัยทั่วไป

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บทคัดย่อ

ผลการศึกษาหลักของดุษฎีนิพนธ์นี้ได้ถูกแบ่งออกเป็นสามส่วนใหญ่ๆ โดยในส่วนแรกนี้ เราแนะนำแนวคิดใหม่ของการหดตัวแบบอาร์ในปริภูมิเมตริก และพิสูจน์ทฤษฎีบทจุดตรึงสำหรับการหดตัวที่วางนัยทั่วไปในปริภูมิเมตริกได้ให้ด้วยอย่างเพื่อสนับสนุนทฤษฎีของเราว่าสำหรับส่วนที่สองนี้ เราได้พิสูจน์ทฤษฎีบทจุดร่วมสำหรับการส่งหดตัวแบบเกรจที่ในปริภูมิเมตริกได้ให้ด้วยอย่างเพื่อสนับสนุนผลลัพธ์ที่ได้รับจากนี้ เราจึงได้พิจารณาจุดร่วมที่สุดซึ่งเป็นผลมาจากการหดตัวหลักของเราว่า เราได้แสดงอย่างเพื่อสนับสนุนผลลัพธ์นั้น ในส่วนสุดท้ายเราได้สร้างหลักการหดตัวร่วมคู่สำหรับการส่งหดตัวแบบแอลฟา เกเรจริงในปริภูมิเมตริกได้ให้ด้วยอย่างเพื่อสนับสนุนผลลัพธ์ที่เราได้ยืนยันในอักษร์ของเราว่า
The main result of this dissertation is divided into three parts. In the first part, we introduce a new concept of $R$-contractions in $b$-metric spaces and prove some fixed point theorems for such contractions in $b$-metric spaces and give examples to support our results. For the second part, we prove some coincidence theorems for Geraghty-type contraction mappings in partially ordered JS-metric spaces. Moreover, we obtain a coupled coincidence point result as a corollary of our main theorem, and we give an example to support our results. For the last part, we establish a coupled coincidence point theorem for $\alpha$-Geraghty contraction type mappings in partially ordered JS-metric spaces. Finally, suitable example is presented to support our main result.
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LIST OF SYMBOLS

\( \in \) Element of a set
\( \subseteq \) Subset of a set
\( \mathbb{R} \) The set of real numbers
\( \| \) Absolute
\( \limsup \) Limit superior
\( \liminf \) Limit inferior
\( \preceq \) Partial order
\( \cap \) Intersection
\( \mathbb{N} \) The set of natural numbers
\( \times \) Cartesian product
\( \text{ran}(d) \) Range of a function \( d \)
\( \infty \) Infinity
\( \max \) Maximum of a set
\( \ln \) Logarithm to the base of a number natural
\( f^{-1} \) Inverse of function \( f \)
\( \implies \) Implies
\( \iff \) If and only if
\( \sup \) Supremum (least upper bound)
\( \sum \) Sum operator
\( \prod \) Product operator
ข้อความแห่งการริเริ่ม

ข้าพเจ้าขอรับรองว่าดุษฎีนิพนธ์เล่มนี้เป็นผลงานของข้าพเจ้า ซึ่งไม่มีส่วนหนึ่งส่วนใดละเมิดลิขสิทธิ์และทรัพย์สินทางปัญญาของผู้อื่น ผลงานการวิจัยนี้ไม่ได้รับการคัดพิมพ์หรือเขียนโดยบุคคลอื่นมาก่อนยังแม้ในระหว่างการดำเนินการวิจัยด้วยลิขสิทธิ์และทรัพย์สินทางปัญญาของผู้อื่น ผลงานการวิจัยนี้ไม่เคยถูกใช้หรือส่งเสริมในการศึกษาในระดับการศึกษาใดๆ ของมหาวิทยาลัยเชียงใหม่หรือสถาบันการศึกษาอื่นมาก่อน.
STATEMENT OF ORIGINALITY

I hereby certify that I am the author of this dissertation. To the best of my knowledge, there are not any parts of this research infringing anyone's copyright and intellectual property. The dissertation does not contain any materials previously written or published by other people except appropriate references for the sake of completeness, I have acknowledged the source in all instances. I also certify that this thesis has not been accepted for the award of any other degree at Chiang Mai University or any other academic institutions.
CHAPTER 1

Introduction

The well-known Banach contraction principle assures the existence and uniqueness of fixed points of certain self-maps in metric spaces. This principle can be applied in various fields such as engineering, economics and computer science. Because of its wide applications, several researchers have extended, improved and generalized the result in many directions.

In 2015, Khojasteh et al. [22] introduced the notion of $Z$-contractions by using a specific function, called simulation function. Then, Khojasteh et al. proved a new fixed point theorem concerning $Z$-contractions. Recently, Roldán López de Hierro and Shahzad [32] introduced the concept of $R$-contraction by using $R$-functions in order to generalize the previous results.

On the other hand, Bakhtin [5] and Czerwik [9] developed the notion of a $b$-metric space and established some fixed point theorems in $b$-metric spaces. Subsequently, several results appeared in this direction [23, 24, 26–28, 33]. Recently, Mongkolkeha et al. [25] introduced the notion of a simulation function in the setting of $b$-metric spaces.

**Question 1.** How can we define some generalizations of $R$-contraction in $b$-metric spaces and prove the existence and uniqueness of fixed point for such classes of mappings in complete $b$-metric spaces?

In the last decades, several authors have worked on generalizations of standard metric spaces. In 2015, Jleli and Samet [17] defined a new class of generalized metric spaces (for short JS-metric spaces). The class of such metric spaces is larger than the class of standard metric spaces and $b$-metric spaces. They proved Banach contraction principle and Ćirić’s fixed point theorem in such spaces. Moreover, Banach contraction principle in generalized metric spaces with partial order is presented. In 1973, Geraghty [13] presented a fixed point theorem for Geraghty contraction mappings in metric spaces.

**Question 2.** How can we prove some coincidence point results for Geraghty-type contraction mappings in partially ordered generalized metric spaces?

The result of Geraghty has been attracting a number of authors [3, 11, 19, 20]. In 2013, Cho, Bae and Karapinar [8] defined the notion of $\alpha$-Geraghty contraction type mapping.
The coupled fixed point was put into use in 1987 by Guo and Lakshmikantham [14]. Later, Bhaskar and Lakshmikatham [7] defined the concept of mixed monotone property and established the existence of a coupled fixed point under the mixed monotone property and applied to a periodic boundary valued problem. In 2008, Radenović [29] extended the results of Bhaskar and Lakshmikatham [7] by using monotone property. In 2015, Kadelburg et al. [19] have studied some coupled coincidence point results for Geraghty-type contraction mappings by using $g$-monotone property in complete partially ordered metric spaces.

**Question 3.** How can we prove some coupled coincidence point theorems for $\alpha$-Geraghty contraction type mappings in partially ordered JS-metric spaces?
CHAPTER 2

Preliminaries

2.1 Background of Fixed Point Theory

Let $X$ be a nonempty set and $T : X \to X$ be a mapping. We say that $x \in X$ is a fixed point of $T$ if and only if $x = Tx$. The set of all fixed points of $T$ is denoted by $F(T)$, i.e., $F(T) = \{x \in X : Tx = x\}$.

Example 2.1.1. If $T : \mathbb{R} \to \mathbb{R}$ is defined on the real numbers by

$$T(x) = x^2 - 3x + 4$$

then $F(T) = \{2\}$.

Definition 2.1.2. [21] Let $X$ be a nonempty set and $d : X \times X \to [0, +\infty)$ be a mapping satisfying the following conditions, for any $x, y, z \in X$:

$(d_1)$ $d(x, y) = 0$ if and only if $x = y$,

$(d_2)$ $d(x, y) = d(y, x)$,

$(d_3)$ $d(x, y) \leq d(x, z) + d(z, y)$.

Then $d$ is called a distance or metric on $X$, and $X$ together with $d$ is called a metric space which will be denoted by $(X, d)$.

Example 2.1.3. The following are examples of metric spaces:

1. Let $X = \mathbb{R}$. Define a mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x - y| \quad \text{for all } x, y \in X,$$

then $d$ is a metric.

2. Let $X = \mathbb{R}^2$. Define a mapping $d : X \times X \to [0, \infty)$ by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

for all $(x_1, y_1), (x_2, y_2) \in X$, then $d$ is a metric.
Definition 2.1.4. [21] A sequence $\{x_n\}$ in a metric space $(X, d)$ is said to be *convergent* if there exists a point $x \in X$ such that $\lim_{n \to \infty} d(x_n, x) = 0$. In this case, we write either $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

Definition 2.1.5. [21] A sequence $\{x_n\}$ in a metric space $(X, d)$ is said to be *Cauchy sequence* if $\lim_{m,n \to \infty} d(x_m, x_n) = 0$.

Definition 2.1.6. [21] A metric space $(X, d)$ is said to be *complete* if every Cauchy sequence in $X$ converges to a point in $X$.

Definition 2.1.7. [21] A mapping $T : X \to Y$ of a metric space $(X, d)$ into a metric space $(Y, \tilde{d})$ is *continuous* at a point $x_0 \in X$ if for all $x_n$ in $(X, d)$ such that $x_n \to x_0$ then $T(x_n) \to T(x_0)$. A mapping $T$ is *continuous* if it is continuous at each $x$ in $X$.

Definition 2.1.8. [34] Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. Then $T$ is called a *contraction mapping* if there exists a nonnegative number $k < 1$ such that

$$d(T(x), T(y)) \leq kd(x, y), \quad \text{for all } x, y \in X.$$ 

Example 2.1.9. Let $X = \mathbb{R}$ and let $d(x, y) = |x - y|$ for all $x, y \in X$. We define a mapping $T : X \to X$ by

$$T(x) = \frac{1}{3}x \quad \text{for all } x \in X.$$ 

Assume $k = \frac{1}{2}$, then $T$ is a contraction mapping.

In 1922, Banach (1892-1945, Poland) proved a well-known theorem, Banach contraction principle.

Theorem 2.1.10. [34] Let $(X, d)$ be a complete metric space and $T : X \to X$ be a contraction mapping. Then $T$ has a unique fixed point $z$ in $X$. Moreover, for each $x \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for each $n \geq 0$ converges to the fixed point $z$.

In 1973, Geraghty [13] introduced the concept of Geraghty’s contraction mappings. Let $\mathcal{F}$ be the family of all functions $\beta : [0, +\infty) \to [0, 1)$ such that for each $\{t_n\} \subseteq [0, +\infty)$ and $\lim_{n \to \infty} \beta(t_n) = 1$, we have $\lim_{n \to \infty} t_n = 0$.

Example 2.1.11. Let $\beta : [0, +\infty) \to [0, 1)$ be a mapping defined by

$$\beta(t) = \begin{cases} 
\frac{1}{1+t} & \text{if } t \in (0, \infty) \\
0 & \text{if } t = 0.
\end{cases}$$ 

Then $\beta \in \mathcal{F}$. 

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In order to generalize the Banach contraction principle, Geraghty proved the following:

**Definition 2.1.12.** [13] Let \((X, d)\) be a metric space and \(T : X \to X\) be a mapping. Then \(T\) is called a *Geraghty’s contraction* if there exists \(\beta \in \mathcal{F}\) such that

\[
d(T(x), T(y)) \leq \beta(d(x, y))d(x, y), \quad \text{for all } x, y \in X.
\]

**Example 2.1.13.** Let \(X = [0, \infty)\) and let \(d(x, y) = |x - y|\) for all \(x, y \in X\). We define a mapping \(T : X \to X\) by

\[
Tx = \frac{x}{x+1} \quad \text{for all } x \in X.
\]
Assume \(\beta(t) = \frac{1}{1+t}\) for all \(t \in (0, +\infty)\) and \(\beta(0) = 0\). Then \(T\) is a Geraghty’s Contraction.

**Theorem 2.1.14.** [13] Let \((X, d)\) be a complete metric space, and let \(T : X \to X\) be a Geraghty’s contraction. Then \(T\) has a unique fixed point \(z \in X\). Moreover, for each \(x \in X\), the sequence \(\{x_n\}\) defined by \(x_{n+1} = Tx_n\) for each \(n \geq 0\) converges to the fixed point \(z\).

**Definition 2.1.15.** A binary relation, \(\preceq\), on a set \(X\) is a *partial order* if it satisfies the following condition, for any \(x, y, z \in X\):

(i) Reflexivity: \(x \preceq x\),

(ii) Anti-symmetry: If \(x \preceq y\) and \(y \preceq x\) then \(x = y\),

(iii) Transitivity: if \(x \preceq y\) and \(y \preceq z\) then \(x \preceq z\).

If \(\preceq\) is a partial order on \(X\), we say that the pair \((X, \preceq)\) is a *partially ordered set*.

**Example 2.1.16.** The following are examples of a partially order set:

1. The real numbers ordered by the standard less-than-or-equal relation \((\leq)\).

2. The set of subsets of a given set (its power set) ordered by inclusion. The set of sequences ordered by subsequence. The set of strings ordered by substring.

In 2010, Amini-Harandi and Emami [4] extended Theorem 2.1.14 to the setting of partially ordered metric spaces:

**Theorem 2.1.17.** Let \((X, d, \preceq)\) be a complete partially ordered metric space. Let \(T : X \to X\) be a mapping satisfying the following conditions:

(i) \(T\) is increasing (i.e., if for all \(x, y \in X\) such that \(x \preceq y\) then \(f(x) \preceq f(y)\)),

(ii) ...
(ii) there exists \( x_0 \in X \) such that \( x_0 \leq T(x_0) \),

(iii) there exists \( \beta \in \mathcal{F} \) such that if for all \( x, y \in X \) such that \( x \preceq y \) then

\[
d(T(x), T(y)) \leq \beta(d(x, y))d(x, y),
\]

(iv) either \( T \) is continuous or for any nondecreasing sequence \( \{x_n\} \) in \( X \), if \( x_n \to x \in X \) then \( x_n \preceq x \) for all \( n \geq 1 \).

Then \( T \) has a unique fixed point in \( X \).

### 2.2 \( \mathcal{Z} \)-Contraction and \( \mathcal{R} \)-Contraction

In 2015, Khojasteh et al. [22] introduced a simulation function as shown below:

**Definition 2.2.1.** [22] A simulation function is a mapping \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) satisfying the following conditions:

\[
(\zeta_1) \quad \zeta(0, 0) = 0;
\]

\[
(\zeta_2) \quad \zeta(t, s) < s - t, \text{ for all } t, s > 0;
\]

\[
(\zeta_3) \quad \text{if } \{t_n\}, \{s_n\} \text{ are sequences in } [0, \infty) \text{ such that } \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n > 0, \text{ then } \lim_{n \to \infty} \zeta(t_n, s_n) < 0.
\]

The class of all simulation functions \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) is denoted by \( \mathcal{Z} \).

**Example 2.2.2.** The following are examples of simulation functions given by Khojasteh [22]:

1. Let \( \lambda \in \mathbb{R} \) be such that \( \lambda < 1 \) and define a mapping \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by

\[
\zeta(t, s) = \lambda s - t \quad \text{for all } s, t \in [0, \infty).
\]

Then \( \zeta \in \mathcal{Z} \).

2. Define a mapping \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by \( \zeta(t, s) = \psi(s) - \phi(t) \) for all \( t, s \in [0, \infty) \), where \( \psi, \phi : [0, \infty) \to [0, \infty) \) are two continuous functions such that \( \psi(t) = \phi(t) = 0 \) if and only if \( t = 0 \) and \( \psi(t) < t \leq \phi(t) \) for all \( t > 0 \). Then \( \zeta \in \mathcal{Z} \).
Definition 2.2.3. [22] Let \((X, d)\) be a metric space, \(T : X \to X\) a mapping and \(\zeta \in \mathcal{Z}\). Then \(T\) is called a \(\mathcal{Z}\)-contraction with respect to \(\zeta\) if the following condition is satisfied

\[
\zeta(d(Tx, Ty), d(x, y)) \geq 0 \text{ for all } x, y \in X.
\]

Note that \(\mathcal{Z}\)-contraction generalizes Geraghty contraction by defining \(\zeta(t, s) = s\beta(s) - t\) for all \(t, s \in [0, \infty)\), where \(\beta \in \mathcal{F}\).

Example 2.2.4. Let \(X = [0, 1]\) and \(d : X \times X \to \mathbb{R}\) be defined by \(d(x, y) = |x - y|\). Define a mapping \(T : X \to X\) such that

\[
Tx = \frac{x}{3} \text{ for all } x \in X.
\]

Define a mapping \(\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}\) such that \(\zeta(t, s) = \frac{1}{2}s - t\) for all \(t, s \in [0, \infty)\).

Then \(T\) is a \(\mathcal{Z}\)-contraction with respect to \(\zeta\).

Theorem 2.2.5. [22] Let \((X, d)\) be a complete metric space, and let \(T : X \to X\) be a \(\mathcal{Z}\)-contraction with respect to \(\zeta\). Then \(T\) has a unique fixed point \(z\) in \(X\). Moreover, for each \(x \in X\), the sequence \(\{x_n\}\) defined by \(x_{n+1} = Tx_n\) for each \(n \geq 0\) converges to the fixed point \(z\).

Example 2.2.6. [22] Let \(X = [0, 1]\) and \(d : X \times X \to \mathbb{R}\) be defined by \(d(x, y) = |x - y|\). Define a mapping \(T : X \to X\) such that

\[
Tx = \frac{x}{x + 1} \text{ for all } x \in X.
\]

Define a mapping \(\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}\) such that \(\zeta(t, s) = \frac{s}{s+1} - t\) for all \(t, s \in [0, \infty)\).

Therefore, all the conditions of Theorem 2.2.5 are satisfied and \(T\) has a unique fixed point \(z = 0 \in X\).

In 2015, Roldán López de Hierro and Shahzad [32] introduced \(R\)-functions and \(R\)-contractions as shown below:

Definition 2.2.7. [32] Let \(A \subseteq \mathbb{R}\) be a nonempty subset. A function \(\varrho : A \times A \to \mathbb{R}\) is called \(R\)-function if it satisfies the following two conditions:

\((g_1)\) If \(\{a_n\} \subset (0, \infty) \cap A\) is a sequence such that \(\varrho(a_{n+1}, a_n) > 0\) for all \(n \in \mathbb{N}\), then \(a_n \to 0\).

\((g_2)\) If \(\{a_n\}, \{b_n\} \subset (0, \infty) \cap A\) are two sequences converging to the same limit \(L \geq 0\) with \(L < a_n\) and \(\varrho(a_n, b_n) > 0\) for all \(n \in \mathbb{N}\), then \(L = 0\).
The class of all \( R \)-functions \( \varrho : A \times A \to \mathbb{R} \) is denoted by \( R_A \). They also considered the following property.

\((g_3)\) If \( \{a_n\}, \{b_n\} \subset (0, \infty) \cap A \) are two sequences such that \( b_n \to 0 \) and \( \varrho(a_n, b_n) > 0 \) for all \( n \in \mathbb{N} \), then \( a_n \to 0 \).

In [32], the authors showed that every simulation function was an \( R \)-function that satisfied \((g_3)\), but the converse was not true.

**Example 2.2.8.** [35] Let \( \varrho_i : [0, \infty) \times [0, \infty) \to \mathbb{R} \) for all \( i = 1, 2, 3 \) be the function defined, for all \( t, s \in [0, \infty) \), by

\[
\varrho_i(t, s) = \begin{cases} 
\frac{s}{t+1} - t & \text{if } i = 1, \\
\frac{s}{t} - t & \text{if } i = 2, \\
\ln(s+1) - t & \text{if } i = 3,
\end{cases}
\]

Then \( \varrho_1, \varrho_2, \varrho_3 \in R_A \).

**Lemma 2.2.9.** [32] Every simulation function is a \( R \)-function that also verifies \((g_3)\).

**Definition 2.2.10.** [32] Let \((X, d)\) be a metric space. A mapping \( T : X \to X \) is called \( R \)-contraction if there exists an \( R \)-function \( \varrho : A \times A \to \mathbb{R} \) such that \( \text{ran}(d) \subseteq A \) and

\[\varrho(d(Tx, Ty), d(x, y)) > 0 \text{ for all } x, y \in X \text{ such that } x \neq y.\]

Notice that if we take \( \varrho(t, s) = ks - t \) for all \( s, t \geq 0 \) and \( k \in [0, 1) \) in Definition 2.2.10, then \( R \)-contraction becomes Banach contraction mapping.

**Example 2.2.11.** Let \( X = [0, 1] \) and \( d : X \times X \to \mathbb{R} \) be defined by \( d(x, y) = |x - y| \). Define a mapping \( T : X \to X \) such that

\[Tx = \frac{x}{2} + 1 \text{ for all } x \in X.\]

Define a mapping \( \varrho : [0, \infty) \times [0, \infty) \to \mathbb{R} \) such that \( \varrho(t, s) = \frac{s}{t} - t \) for all \( t, s \in [0, \infty) \). Then \( T \) is a \( R \)-contraction.

**Theorem 2.2.12.** [32] Let \((X, d)\) be a complete metric space and let \( T : X \to X \) be an \( R \)-contraction with respect to \( \varrho \in R_A \). Assume that, at least, one of the following conditions holds:

(a) \( T \) is continuous.

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(b) The function $\varphi$ satisfies condition $(\varphi_3)$.

(c) $\varphi(t, s) \leq s - t$ for all $t, s \in (0, \infty) \cap A$.

Then $T$ has a unique fixed point $z$ in $X$. Moreover, for each $x \in X$, the sequence \{${x}_n$\} defined by $x_{n+1} = Tx_n$ for each $n \geq 0$ converges to the fixed point $z$.

2.3 $b$-metric

In 1993, Czerwik [9] introduced a $b$-metric space as shown below:

**Definition 2.3.1.** [9] A $b$-metric on a set $X$ is a mapping $d : X \times X \to [0, +\infty)$ satisfying the following conditions: for any $x, y, z \in X$,

(b1) $d(x, y) = 0$ if and only if $x = y$;

(b2) $d(x, y) = d(y, x)$;

(b3) there exists $K \geq 1$ such that $d(x, y) \leq K(d(x, z) + d(z, y))$.

Then $(X, d)$ is known as a $b$-metric space with coefficient $K$.

Note that every metric space is a $b$-metric space with $K = 1$. Some examples of $b$-metric spaces are given below:

**Example 2.3.2.**

1. Let $X = [0, 1]$. Define a mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = (x - y)^2$$

for all $x, y \in X$.

Then $(X, d)$ is a $b$-metric space with coefficient $K = 2$.

2. Let $X = \{1, 2, 3\}$. Define a mapping $d : X \times X \to [0, \infty)$ by $d(1, 1) = d(2, 2) = d(3, 3) = 0$, $d(1, 2) = d(2, 1) = 2$, $d(2, 3) = d(3, 2) = 1$ and $d(1, 3) = d(3, 1) = 6$.

Then $(X, d)$ is a $b$-metric space with coefficient $K = 2$.

3. Let $X = \mathbb{R} \setminus \{0\}$ and define a mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \left|\frac{1}{x} - \frac{1}{y}\right|^2$$

for all $x, y \in X$.

Then $d$ is a $b$-metric with coefficient $K = 2$.

In 2017, Mongkolkeha et al. [25] introduced a simulation function in the framework of $b$-metric spaces shown below:
Definition 2.3.3. [25] Let $K$ be a given real number such that $K \geq 1$. A $K$-simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

$(\zeta_1)$ $\zeta(0,0) = 0$;

$(\zeta_2)$ $\zeta(Kt,s) \leq s - Kt$, for all $t,s > 0$;

$(\zeta_3)$ if $\{t_n\}, \{s_n\}$ are sequences in $[0, \infty)$ such that $\limsup_{n \to \infty} Kt_n = \limsup_{n \to \infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then

$$\limsup_{n \to \infty} \zeta(Kt_n, s_n) < 0.$$ 

The class of all $K$-simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is denoted by $Z^*$.

Example 2.3.4. [25] Let $\lambda, K \in \mathbb{R}$ be such that $\lambda < 1$ and $K \geq 1$. Define the mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(Kt,s) = \begin{cases} s - Kt & \text{if } s < t, \\ \lambda s - Kt & \text{otherwise.} \end{cases}$$

Then $\zeta \in Z^*$ but $\zeta \notin Z$.

Definition 2.3.5. [25] Let $(X,d)$ be a $b$-metric space with $K \geq 1$, $T : X \to X$ be a mapping and $\zeta \in Z^*$. Then $T$ is called a $K$-simulation contraction with respect to $\zeta$ if the following condition is satisfied

$$\zeta(Kd(Tx,Ty),d(x,y)) \geq 0 \text{ for all } x, y \in X.$$ 

Example 2.3.6. Let $X = [0,1]$, $K = 2$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x - y|$. Define a mapping $T : X \to X$ such that

$$Tx = \frac{x}{2} \text{ for all } x \in X.$$ 

Define a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that $\zeta(Kt,s) = \frac{s - Kt}{1 + Ks}$ for all $t,s \in [0, \infty)$. Then $T$ is a $K$-simulation contraction with respect to $\zeta$.

Motivated and inspired by these facts, we define a generalization of $R$-contraction [Definition 2.2.10] in $b$-metric spaces and prove the existence and uniqueness of fixed point for such classes of mappings in complete $b$-metric spaces. This generalizes Theorem 2.2.5 and Theorem 2.2.12.
2.4 α-Geraghty Contraction type

The result of Geraghty has been attracting a number of authors [3, 11, 19, 20]. In 2013, Cho, Bae and Karapinar [8] defined the notion of α-Geraghty contraction type mapping as follows:

**Definition 2.4.1.** [8] Let \((X, d)\) be a metric space and \(\alpha : X \times X \to \mathbb{R}\). A mapping \(T : X \to X\) is called \(\alpha\)-Geraghty contraction type if there exits \(\beta \in \mathcal{F}\) such that for all \(x, y \in X\),

\[
\alpha(x, y)d(Tx, Ty) \leq \beta(d(x, y))d(x, y).
\]

**Example 2.4.2.** Let \(X = [0, 1]\), and let \(d(x, y) = |x - y|\) for all \(x, y \in X\). Let \(\beta(t) = \frac{1}{1+t}\) for all \(t \geq 0\). We define a mapping \(T : X \to X\) by

\[
Tx = 3x \quad \text{for all } x \in X.
\]

Let a function \(\alpha : X \times X \to [0, \infty)\) by \(\alpha(x, y) = \frac{1}{9}\) for all \(x, y \in X\). Then \(T\) is \(\alpha\)-Geraghty contraction type.

**Definition 2.4.3.** [8] Let \(T : X \to X\) be a mapping and \(\alpha : X \times X \to \mathbb{R}\) be a function. Then \(T\) is said to be \(\alpha\)-admissible if for all \(x, y \in X\)

\[
\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.
\]

**Definition 2.4.4.** [8] An \(\alpha\)-admissible map \(T\) is said to be triangular \(\alpha\)-admissible if for all \(x, y, z \in X\)

\[
\alpha(x, z) \geq 1 \text{ and } \alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1.
\]

**Example 2.4.5.** Let \(X = [0, \infty)\), and let a mapping \(T : X \to X\) by define \(T(x) = \frac{1}{3}x\) for all \(x \in X\). Define a function \(\alpha : X \times X \to [0, \infty)\) by

\[
\alpha(x, y) = \begin{cases} 
1 & \text{if } 0 \leq x, y \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Then, \(T\) is \(\alpha\)-admissible and triangular \(\alpha\)-admissible.

**Theorem 2.4.6.** [8] Let \((X, d)\) be a complete metric space, \(\alpha : X \times X \to \mathbb{R}\) be a function, and let \(T : X \to X\) be a mapping. Suppose that the following conditions are satisfied:

(i) \(T\) is a \(\alpha\)-Geraghty contraction type;

(ii) \(T\) is triangular \(\alpha\)-admissible;
(iii) there exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq 1 \);

(v) either \( T \) is continuous or

\[
\liminf_{n \to \infty} \alpha(x_n, x) > 0
\]

for any cluster point \( x \) of a sequence \( \{x_n\} \) with \( \alpha(x_n, x_{n+1}) \geq 1 \).

Then \( T \) has a fixed point \( z \in X \). Moreover, for each \( x \in X \), the sequence \( \{x_n\} \) defined by \( x_{n+1} = Tx_n \) for each \( n \geq 0 \) converges to the fixed point \( z \).

Example 2.4.7. [8] Let \( X = [0, \infty) \), and let \( d(x, y) = |x - y| \) for all \( x, y \in X \). Let \( \beta(t) = \frac{1}{1+t} \) for all \( t \geq 0 \). We define a mapping \( T : X \to X \) by

\[
Tx = \begin{cases} 
\frac{1}{2}x & \text{if } 0 \leq x \leq 1, \\
2x & \text{if } x > 1 
\end{cases}
\]

and a function \( \alpha : X \times X \to [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases} 
1 & \text{if } 0 \leq x, y \leq 1, \\
0 & \text{otherwise.}
\end{cases}
\]

Then, \( T \) has a fixed point \( z \), that is \( z = 0 \).

Note that: A mapping \( T \) in Example 2.4.7 does not satisfy Geraghty’s contraction. In fact, for \( x = 1, y = 2 \), we have

\[
d(T1, T2) = d(\frac{1}{2}, 4) = \frac{7}{2} > \frac{1}{2} = \frac{1}{2}d(1, 2) = \beta(d(1, 2))d(1, 2).
\]

2.5 Coincidence Point and Coupled Coincidence Point

Let \( X \) be a nonempty set. Let \( f, g : X \to X \) be a pair of self-maps on \( X \).

(i) We say that \( x \in X \) is a coincidence point of \( f \) and \( g \) if \( f(x) = g(x) \).

(ii) We say that \( x \in X \) is a common fixed point of \( f \) and \( g \) if \( x = f(x) = g(x) \).

(iii) We say \( f \) and \( g \) are commuting if \( g(f(x)) = f(g(x)) \), for each \( x \in X \).

(iv) For a partial order \( \preceq \), \( E_\preceq := \{(x, y) \in X \times X : x \preceq y\} \).

Example 2.5.1. Let \( X = \mathbb{R} \) and define mapping \( f, g : X \to X \) by

\[
f(x) = x^2, \quad g(x) = x^3 \quad \text{for all } x \in X.
\]

Then \( x = 0 \) and \( x = 1 \) are coincidence point and common fixed point of \( f \) and \( g \).
**Definition 2.5.2.** Let $X$ be a nonempty set, $F : X \times X \to X$ and $g : X \to X$. We say that an element $(x, y) \in X \times X$ is a coupled coincidence point of $g$ and $F$ if

$$gx = F(x, y) \text{ and } gy = F(y, x).$$

We say that $g$ and $F$ are **commuting** if

$$gF(x, y) = F(gx, gy), \text{ for every } x, y \in X.$$

**Example 2.5.3.** Let $X = \mathbb{R}$, we define mapping $F : X \times X \to X$ and $g : X \to X$ defined by

$$F(x) = \frac{x + y}{3}, \quad g(x) = 2x \quad \text{for all } x, y \in X.$$

Then $(x, y) = (0, 0)$ is coupled coincidence point of $g$ and $F$.

**Definition 2.5.4.** ([19]) Let $(X, \preceq)$ be a partially ordered set. Let $F : X \times X \to X$ and $g : X \to X$ be a pair of mappings. Then $F$ has the $\preceq$-$g$-monotone property if and only if for every $x, y \in X$,

$$x_1, x_2 \in X, \quad (g(x_1), g(x_2)) \in E_\preceq \implies (F(x_1, y), F(x_2, y)) \in E_\preceq,$$

and,

$$y_1, y_2 \in X, \quad (g(y_1), g(y_2)) \in E_\preceq \implies (F(x, y_1), F(x, y_2)) \in E_\preceq.$$

**Example 2.5.5.** Let $X = [0, 1] \subseteq \mathbb{R}$ and $(X, \preceq)$ be a partially ordered set. We define mapping $F : X \times X \to X$ and $g : X \to X$ defined by

$$F(x) = \ln(1 + x + y), \quad g(x) = 2x \quad \text{for all } x, y \in X.$$

Then $F$ has the $\preceq$-$g$-monotone property.

**Definition 2.5.6.** ([19]) Let $(X, d)$ be a metric space and let $g : X \to X$, $F : X \times X \to X$. The mappings $g$ and $F$ are said to be **compatible** if

$$\lim_{n \to \infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

and,

$$\lim_{n \to \infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} g x_n$ and $\lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} g y_n$. 

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Example 2.5.7. Let $X = [0, 1] \subseteq \mathbb{R}$, and let $d(x, y) = |x - y|$ for all $x, y \in X$. We define mapping $F : X \times X \to X$ and $g : X \to X$ by

$$F(x, y) = \ln(1 + \frac{x^2}{2} + \frac{y^2}{2}), \quad gx = x^2, \quad \text{for all } x, y \in X.$$ 

Then $g$ and $F$ are compatible.

Let $\Theta$ be the family of all functions $\theta : [0, +\infty) \times [0, +\infty) \to [0, 1)$ satisfying the following conditions:

$(\theta_1)$ $\theta(s, t) = \theta(t, s)$ for all $s, t \in [0, +\infty),$

$(\theta_2)$ for any two sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative real numbers,

$$\lim_{n \to \infty} \theta(s_n, t_n) = 1 \text{ implies } \lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = 0.$$

Example 2.5.8. [19] The following are examples of some functions belonging to $\Theta$.

1. $\theta(s, t) = k$ for all $s, t \in [0, \infty)$, where $k \in [0, 1)$.

2. $\theta(s, t) = \begin{cases} \frac{\ln(1 + ks + lt)}{ks + lt}, & \text{if } s > 0 \text{ or } t > 0, \\
 r \in [0, 1), & \text{if } s = 0 \text{ and } t = 0, \end{cases}$ where $k, l \in (0, 1)$.

3. $\theta(s, t) = \begin{cases} \frac{\ln(1 + \max\{s, t\})}{\max\{s, t\}}, & \text{if } s > 0 \text{ or } t > 0, \\
 r \in [0, 1), & \text{if } s = 0 \text{ and } t = 0. \end{cases}$

In 2015, Kadelburg et al. [19] have studied some coupled coincidence point results for Geraghty-type contraction mappings by using $\preceq$-g-monotone property in complete partially ordered metric spaces.

Theorem 2.5.9. [19] Let $(X, d, \preceq)$ be a complete partially ordered metric space, $F : X \times X \to X$ and $g : X \to X$. Suppose that the following conditions hold:

(i) $F(X^2) \subseteq g(X),$

(ii) $F$ has the $\preceq$-g-monotone property,

(iii) there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \preceq F(y_0, x_0),$

(iv) there exists $\theta \in \Theta$ such that

$$d(F(x, y), F(u, v)) \leq \theta(d(gx, gu), d(gy, gv)) \max\{d(gx, gu), d(gy, gv)\},$$

for all $x, y, u, v \in X$ with $(gx, gu), (gy, gv) \in E_{\preceq}$ or $(gu, gx), (gv, gy) \in E_{\preceq},$
(v) $g$ and $F$ are compatible,

(vi) $g$ is continuous and $g(X)$ is closed,

(vii) (a) $F$ is continuous or (b) if for an increasing sequence $\{x_n\}$ in $X$, $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then, $g$ and $F$ have a coupled coincidence point.

**Example 2.5.10.** [19] Let $X = [0, 1] \subseteq \mathbb{R}$, and let $d(x, y) = |x - y|$ for all $x, y \in X$. We define mapping $g : X \to X$ and $F : X \times X \to X$ by

$\quad gx = x^2, \quad F(x, y) = \ln(1 + \frac{x^2}{2} + \frac{y^2}{2})$, for all $x, y \in X$.

Let $\theta \in \Theta$ be defined by

$\theta(s, t) = \begin{cases} 
\ln\left(1 + \frac{s}{2} + \frac{t}{2}\right), & \text{if } s > 0 \text{ or } t > 0, \\
\frac{s}{2} + \frac{t}{2}, & \text{if } s = 0 \text{ and } t = 0.
\end{cases}$

Then, $g$ and $F$ have coupled coincidence point which is $(0, 0)$.

### 2.6 JS-metric spaces

A new class of generalized metric spaces was introduced by Jleli and Samet [17] (for short JS-metric spaces). The class of such metric spaces is covering the class of standard metric spaces and $b$-metric spaces. They proved Banach contraction principle and Ćirić’s fixed point theorem in such spaces. We recall the definition of a JS-metric space. Let $X$ be a nonempty set and $D : X \times X \to [0, +\infty]$ be a given mapping. For every $x \in X$, Jleli and Samet [17] defined the following set:

$C(D, X, x) = \{\{x_n\} \subseteq X : \lim_{n \to \infty} D(x_n, x) = 0\}$.

**Definition 2.6.1.** [17] A JS-metric on a set $X$ is a mapping $D : X \times X \to [0, +\infty]$ satisfying the following conditions:

(D$_1$) for any $x, y \in X$, if $D(x, y) = 0$, then $x = y$,

(D$_2$) $D(x, y) = D(y, x)$ for all $x, y \in X$,

(D$_3$) there exists $K > 0$ such that

\[
\text{if } x, y \in X \text{ and } \{x_n\} \in C(D, X, x), \text{ then } D(x, y) \leq K \limsup_{n \to \infty} D(x_n, y).
\]
Then \((X, D)\) is called a \textit{JS-metric space}.

**Example 2.6.2.**

1. Let \(X = \{0, 1\}\). Define a mapping \(D : X \times X \to [0, \infty)\) by
   \[
   D(x, y) = \begin{cases} 
   0 & \text{if } x = y \\
   \infty & \text{if } x \neq y.
   \end{cases}
   \]
   Then \(D\) is a JS-metric but not a \(b\)-metric.

2. Let \(X = \mathbb{R}\). Define a mapping \(D : X \times X \to [0, \infty)\) by
   \[
   D(x, y) = |x| + |y| \quad \text{for all } x, y \in X.
   \]
   Then \(D\) is JS-metric and \(b\)-metric but not a standard metric.

3. Let \(X = [0, 1]\). Define a mapping \(D : X \times X \to [0, \infty)\) by
   \[
   D(x, y) = \max\{x, y\} \quad \text{for all } x, y \in X.
   \]
   Then \(D\) is a JS-metric but not a \(b\)-metric.

**Definition 2.6.3.** [17] Let \((X, D)\) be a JS-metric space. Let \(\{x_n\}\) be a sequence in \(X\) and \(x \in X\). We say that
   (i) \(\{x_n\}\) \(D\)-converges to \(x\) if \(\{x_n\} \in C(D, X, x)\),
   (ii) \(\{x_n\}\) is a \(D\)-Cauchy sequence if \(\lim_{m,n \to \infty} D(x_n, x_{n+m}) = 0\), and
   (iii) \(X\) is \(D\)-complete if every \(D\)-Cauchy sequence in \(X\) is \(D\)-convergent to some element in \(X\).

**Definition 2.6.4.** Let \((X, D)\) be a JS-metric space. A mapping \(f : X \to X\) is called \textit{continuous at a point} \(x_0 \in X\) if
   \[
   \lim_{n \to \infty} D(f(x_n), f(x_0)) = 0 \quad \text{when ever } \{x_n\} \in C(D, X, x_0).
   \]
   A mapping \(f\) is \textit{continuous} if it is continuous at each \(x\) in \(X\).

**Proposition 2.6.5.** [17] Let \((X, D)\) be a JS-metric space. Let \(\{x_n\}\) be a sequence in \(X\) and \((x, y) \in X \times X\). If \(\{x_n\}\) \(D\)-converges to \(x\) and \(\{x_n\}\) \(D\)-converges to \(y\), then \(x = y\).

Motivated and inspired by these facts, we establish some coincidence point results for Geraghty-type contraction mappings in partially ordered JS-metric spaces.

Finally, we will prove some coupled coincidence point theorems for \(\alpha\)-Geraghty contraction type mappings in partially ordered JS-metric spaces generalizing Theorem 2.5.9.
3.1 Fixed Point Theorems for Generalized R-Contraction in b-Metric Spaces

In this section, we introduce the new family of functions in the setting of b-metric spaces.

Definition 3.1.1. Let $K$ be a given real number such that $K \geq 1$. A function $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called $R'$-function if it satisfies the following two conditions:

1. ($\varrho'1$) If $\{a_n\} \subset (0, \infty)$ is a sequence such that $\varrho(Ka_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

2. ($\varrho'2$) If $\{a_n\}, \{b_n\} \subset (0, \infty)$ are two sequences such that $\limsup_{n \to \infty} Ka_n = \limsup_{n \to \infty} b_n = L \geq 0$ and verifying that $L < Ka_n$ and $\varrho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $L = 0$.

The class of all $R'$-functions $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is denoted by $R'$. We also consider the following property.

3. ($\varrho'3$) If $\{a_n\}, \{b_n\} \subset (0, \infty)$ are two sequences such that $b_n \to 0$ and $\varrho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Example 3.1.2. Let $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

\[
\varrho(2t, s) = \begin{cases} 
\frac{s}{2} - 2t & \text{if } 2t \neq s, \\
0 & \text{if } 2t = s.
\end{cases}
\]

Then $\varrho$ is a $R'$-function that also verifies ($\varrho'3$) but it is not a $K$-simulation function.

Lemma 3.1.3. Every $K$-simulation function is a $R'$-function that also verifies ($\varrho'3$).

Proof. Let $K$ be a given real number such that $K \geq 1$ and $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a $K$-simulation function.

1. ($\varrho'1$) Let $\{a_n\} \subset (0, \infty)$ be a sequence such that $\varrho(Ka_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$.

By condition ($\varrho'2$),

\[0 < \varrho(Ka_{n+1}, a_n) \leq a_n - Ka_{n+1} \leq a_n - a_{n+1},\]
for all $n \in \mathbb{N}$. So \{\(a_n\)\} is a strictly decreasing sequence of positive real numbers. Then \{\(a_n\)\} is convergent, given $L \geq 0$ such that $a_n \to L$. We will show that $L = 0$. By contradiction, assume $L > 0$. Let $t_n = \frac{a_{n+1}}{K}$ and $s_n = a_n$ for all $n \in \mathbb{N}$. By condition (\(\zeta_4'\)),

$$0 \leq \limsup_{n \to \infty} \rho(a_{n+1}, a_n) = \limsup_{n \to \infty} \rho(Kt_n, s_n) < 0,$$

which is a contradiction. Therefore $a_n \to 0$.

(\(\sigma_2'\)) Let \{\(a_n\)\}, \{\(b_n\)\} $\subset (0, \infty)$ be sequences such that $\limsup_{n \to \infty} Ka_n = \limsup_{n \to \infty} b_n = L \geq 0$ and satisfying that $L < Ka_n$ and $\rho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$. We will show that $L = 0$. By contradiction, assume $L > 0$. By condition (\(\zeta_2'\)), $0 < \rho(Ka_n, b_n) \leq b_n - Ka_n$. Then

$$a_n \leq Ka_n < b_n \text{ for all } n \in \mathbb{N}.$$

By condition (\(\zeta_3'\)), $0 \leq \limsup_{n \to \infty} \rho(Ka_n, b_n) < 0$, which is a contradiction. Therefore $L = 0$.

(\(\sigma_3'\)) Let \{\(a_n\)\}, \{\(b_n\)\} $\subset (0, \infty)$ such that $b_n \to 0$ and $\rho(Ka_n, b_n) > 0$ for all $n \in \mathbb{N}$. Since $\rho$ is a $K$-simulation function, $0 < \rho(Ka_n, b_n) \leq b_n - Ka_n$ for all $n \in \mathbb{N}$. Hence $0 < Ka_n < b_n$ for all $n \in \mathbb{N}$, this implies that, $Ka_n \to 0$. Since $K \geq 1$, $a_n \to 0$.

Now, we use $R'$-functions to define a new class of contractions in $b$-metric spaces.

**Definition 3.1.4.** Let $(X, d)$ be a $b$-metric space with coefficient $K \geq 1$ and let $T : X \to X$ be a mapping. We will say that $T$ is a $R'$-contraction if there exists a $R'$-function $\rho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that

$$\rho(Kd(Tx, Ty), d(x, y)) > 0 \text{ for all } x, y \in X \text{ such that } x \neq y.$$

**Example 3.1.5.** Let $X = [0, 1]$ and $d(x, y) = (x - y)^2$ for all $x, y \in X$, then $(X, d)$ is a complete $b$-metric space with coefficient $K = 2$. Let $T : X \to X$ be given by

$$T(x) = \frac{x}{\sqrt{7}} \text{ for all } x \in X.$$

Define $\rho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\rho(2t, s) = \begin{cases} \frac{s}{3} - 2t & \text{if } 2t \neq s, \\ 0 & \text{if } 2t = s. \end{cases}$$

Then $T$ is a $R'$-contraction.
Now, we are ready to give our main theorem.

**Theorem 3.1.6.** Let \((X, d)\) be a complete \(b\)-metric space with coefficient \(K \geq 1\). Let \(T : X \to X\) be \(R\)'-contraction with respect to \(\varrho \in R^*\). If \(\varrho(Kt, s) \leq s - Kt\) for all \(s, t \in (0, \infty)\) then \(T\) has a unique fixed point.

**Proof.** Let \(x_0 \in X\) be an arbitrary point. Let \(\{x_n\}\) be a Picard sequence of \(T\) based on \(x_0\), that is, \(x_{n+1} = Tx_n\). If there exists \(n_0 \in \mathbb{N}\) such that \(x_{n_0+1} = x_{n_0}\), then \(Tx_{n_0} = x_{n_0}\) which implies that \(x_{n_0}\) is a fixed point. Assume \(x_n \neq x_{n+1}\) for all \(n \in \mathbb{N}\). Let \(\{a_n\} \subset (0, \infty)\) be a sequence defined by \(a_n = d(x_n, x_{n+1}) > 0\) for all \(n \in \mathbb{N}\). By \(R\)'-contraction,

\[
\varrho(Ka_{n+1}, a_n) = \varrho(Kd(x_{n+1}, x_{n+2}), d(x_n, x_{n+1})) \\
= \varrho(Kd(Tx_n, Tx_{n+1}), d(x_n, x_{n+1})) \\
> 0.
\]

From the condition \((\varrho_1)\),

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = \lim_{n \to \infty} a_n = 0.
\]

Next, we show that \(\{x_n\}\) is a Cauchy sequence reasoning by contradiction. If \(\{x_n\}\) is not a Cauchy sequence, then there exists \(\varepsilon_0 > 0\) such that

\[
d(x_{n_k}, x_{m_k}) > \varepsilon_0 \quad \text{and} \quad d(x_{n_k}, x_{m_k-1}) \leq \varepsilon_0 \quad \text{for all} \quad m_k > n_k \geq k. \tag{3.1}
\]

Consider

\[
\varepsilon_0 < d(x_{n_k}, x_{m_k}) \leq K(d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k})) \quad \text{for all} \quad k \in \mathbb{N}.
\]

Taking limit superior \(k\) to infinity,

\[
\varepsilon_0 \leq \limsup_{k \to \infty} d(x_{n_k}, x_{m_k}) \leq K\varepsilon_0. \tag{3.2}
\]

Since \(d(x_{n_k-1}, x_{m_k-1}) \leq K(d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k-1}))\) for all \(k \in \mathbb{N}\), taking limit superior \(k\) to infinity,

\[
\limsup_{k \to \infty} d(x_{n_k-1}, x_{m_k-1}) \leq K\varepsilon_0. \tag{3.3}
\]

If \(d(x_{n_k-1}, x_{m_k-1}) = 0\), for some \(k_0 \in \mathbb{N}\) then \(x_{n_k} = x_{m_k}\), which contradict to (3.1). Therefore \(x_{n_k-1} \neq x_{m_k-1}\) for all \(k \in \mathbb{N}\). By \(R\)'-contraction,

\[
0 < \varrho(Kd(x_{n_k}, x_{m_k}), d(x_{n_k-1}, x_{m_k-1})) \leq d(x_{n_k-1}, x_{m_k-1}) - Kd(x_{n_k}, x_{m_k}).
\]

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This implies that

\[ Kd(x_{n_k}, x_{m_k}) < d(x_{n_k-1}, x_{m_k-1}) \quad \text{for all } k \in \mathbb{N}. \quad (3.4) \]

By (3.2), (3.3) and (3.4).

\[ K \varepsilon_0 = \limsup_{k \to \infty} Kd(x_{n_k}, x_{m_k}) \leq \limsup_{k \to \infty} d(x_{n_k-1}, x_{m_k-1}) \leq K \varepsilon_0. \]

That is

\[ \limsup_{k \to \infty} Kd(x_{n_k}, x_{m_k}) = \limsup_{k \to \infty} d(x_{n_k-1}, x_{m_k-1}) = K \varepsilon_0. \]

Since \( K \varepsilon_0 < Kd(x_{n_k}, x_{m_k}) \), for all \( k \in \mathbb{N} \) and the condition \( (\rho'_2) \), \( K \varepsilon_0 = 0 \).

That is a contradiction. Thus \( \{x_n\} \) is a Cauchy sequence. Since \((X, d)\) is complete, there exists \( z \in X \) such that \( x_n \to z \).

Next, we will show that \( z \) is a fixed point reasoning by contradiction. If \( z \) is not a fixed point, that is \( z \neq Tz \). Let \( \varepsilon = \frac{d(z, Tz)}{2} > 0 \). Since \( x_n \to z \), there exists \( N \) such that \( d(x_n, z) < \varepsilon \) for all \( n > N \). \hspace{1cm} (3.5)

Let \( \Omega = \{n \in \mathbb{N} : d(x_n, z) = 0\} \). Assume that \( \Omega \) is not finite, then we can find \( n_0 > N \) such that \( d(x_{n_0}, z) = 0 \) i.e. \( x_{n_0} = z \). By (3.5),

\[ \varepsilon > d(x_{n_0+1}, z) = d(Tx_{n_0}, z) = d(Tz, z), \]

which is a contradiction. Therefore \( \Omega \) is finite, there exists \( n_0 \) such that \( d(x_n, z) > 0 \) for all \( n > n_0 \). Since \( T \) is a \( \rho' \)-contraction,

\[ 0 < \rho(Kd(Tx_n, Tz), d(x_n, z)) \leq d(x_n, z) - Kd(Tx_n, Tz), \quad \text{for all } n > n_0. \]

Hence,

\[ Kd(Tx_n, Tz) < d(x_n, z), \quad \text{for all } n > n_0. \]

Taking limit \( n \) to infinity,

\[ \lim_{n \to \infty} Kd(Tx_n, Tz) \leq \lim_{n \to \infty} d(x_n, z) = 0. \]

Thus \( \lim_{n \to \infty} Kd(Tx_n, Tz) = 0 \), that is, \( \{x_{n+1} = Tx_n\} \to Tz \). By the uniqueness of the limit, \( Tz = z \). Finally, let us show that \( z \) is unique fixed point of \( T \). Assume \( x = Tx \) and \( y = Ty \) such that \( x \neq y \). Let \( a_n = d(x, y) > 0 \) for all \( n \in \mathbb{N} \). Consider

\[ \rho(Ka_{n+1}, a_n) = \rho(Kd(x, y), d(x, y)) = \rho(Kd(Tx, Ty), d(x, y)) > 0. \]

From \( (\rho'_1) \), then \( a_n \to 0 \), which imply that \( d(x, y) = 0 \), which is a contradiction. So \( x = y \).
This following results are immediately true by our main result.

**Corollary 3.1.7.** [22] Let \( (X,d) \) be a complete metric space and \( T : X \to X \) be a \( \zeta \)-contraction with respect to a certain simulation function \( \zeta \), that is,

\[
\zeta(d(Tx,Ty),d(x,y)) \geq 0, \quad \text{for all } x, y \in X.
\]

Then \( T \) has a unique fixed point. Moreover, for every \( x_0 \in X \), the Picard sequence \( \{T^n x_0\} \) converges to this fixed point.

**Corollary 3.1.8.** [18] Let \( (X,d) \) be a complete \( b \)-metric space and let \( T : X \to X \) be a mapping. Suppose that there exists \( \lambda \in (0,1) \) such that

\[
d(Tx,Ty) \leq \lambda d(x,y) \quad \text{for all } x, y \in X.
\]

Then \( T \) has a unique fixed point.

**Corollary 3.1.9.** [31] Let \( (X,d) \) be a complete \( b \)-metric space and let \( T : X \to X \) be a mapping. Suppose that there exists a lower semi-continuous function \( \varphi : [0,\infty) \to [0,\infty) \) with \( \varphi^{-1}(0) = 0 \) such that

\[
d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) \quad \text{for all } x, y \in X.
\]

Then \( T \) has a unique fixed point.

**Proof.** The result follows from Theorem 3.1.6, by taking as simulation function

\[
\zeta(t,s) = s - \varphi(s) - t \quad \text{for all } t, s \geq 0.
\]

**Corollary 3.1.10.** [30] Let \( (X,d) \) be a complete \( b \)-metric space and let \( T : X \to X \) be a mapping. Suppose that there exists a function \( \varphi : [0,\infty) \to [0,1) \) with \( \limsup_{t \to r^+} \varphi(t) < 1 \) for all \( r > 0 \) such that

\[
d(Tx,Ty) \leq \varphi(d(x,y))d(x,y) \quad \text{for all } x, y \in X.
\]

Then \( T \) has a unique fixed point.

**Proof.** The result follows from Theorem 3.1.6, by taking as simulation function

\[
\zeta(t,s) = s\varphi(s) - t \quad \text{for all } t, s \geq 0.
\]
The following examples support our main result.

**Example 3.1.11.** Let $X = [0, 1]$ and $d(x, y) = (x - y)^2$ for all $x, y \in X$, then $(X, d)$ is a complete $b$-metric space with coefficient $K = 2$. Let $T : X \to X$ be given by $T(x) = \frac{x}{\sqrt{10}(2 + x)}$ for all $x \in X$. Define $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$
\varrho(2t, s) = \begin{cases} 
\frac{s}{2} - 2t & \text{if } 2t \neq s, \\
0 & \text{if } 2t = s.
\end{cases}
$$

Therefore, all the requirements of previous Theorem 3.1.6 are satisfied and $x = 0$ is a unique fixed point in $X$.

**Example 3.1.12.** Let $X = [0, 1]$ and $d(x, y) = (x - y)^2$ for all $x, y \in X$, then $(X, d)$ is a complete $b$-metric space with coefficient $K = 2$. Let $T : X \to X$ be given by $T(x) = \frac{x}{\sqrt{10}(2 + x)}$ for all $x \in X$. For all $x, y \in X$ such that $x \neq y$, we have

$$
d(Tx, Ty) = \left( \frac{x}{\sqrt{10}(2 + x)} - \frac{y}{\sqrt{10}(2 + y)} \right)^2
\begin{align*}
&= \frac{2}{5} \frac{x - y}{(2 + x)(2 + y)} \\
&= \frac{2}{5} \frac{(x - y)^2}{(4 + 2x + 2y + xy)^2} \\
&= \frac{2}{5} \frac{(x - y)^2}{(1 + (x - y)^2)^2}.
\end{align*}
$$

Define $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $\varrho(2t, s) = \frac{s}{1 + s} - 2t$, then $\varrho \in R^{\times}$.

Therefore

$$
\varrho(2d(Tx, Ty), d(x, y)) = \frac{(x - y)^2}{1 + (x - y)^2} - 2d(Tx, Ty)
\begin{align*}
&\geq \frac{(x - y)^2}{1 + (x - y)^2} - \frac{4}{5} \left( \frac{(x - y)^2}{1 + (x - y)^2} \right) \\
&= \frac{1}{5} \left( \frac{(x - y)^2}{1 + (x - y)^2} \right) \\
&> 0.
\end{align*}
$$

Therefore, $T$ is a $R^{\times}$-contraction and $\varrho(2t, s) = \frac{s}{1 + s} - 2t \leq s - 2t$ for all $s, t \in (0, \infty)$. By Theorem 3.1.6, $T$ has a unique fixed point, that is, $x = 0$. 

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3.2 Coincidence Point Theorems for Geraghty-type Contraction mappings in JS-Metric Spaces

In this section, we consider the existence of coincidence point for Geraghty-type contraction in the setting of JS-metric spaces. Because of the larger class of spaces, the proof presented here is different from a proof appeared in [13].

Let $F'$ be the family of all functions $\beta : [0, +\infty] \to [0, 1)$ satisfying the condition
\[
\lim_{n \to \infty} \beta(t_n) = 1 \text{ implies } \lim_{n \to \infty} t_n = 0 \text{ for all } t_n \in [0, +\infty].
\]

**Example 3.2.1.** Let $\beta : [0, +\infty] \to [0, 1)$ be a mapping defined by
\[
\beta(t) = \begin{cases} 
\frac{1}{1 + t} & \text{if } t \in (0, \infty) \\
0 & \text{if } t \in \{0, \infty\}.
\end{cases}
\]

Then $\beta \in F'$.

**Definition 3.2.2.** [17] Let $(X, \preceq)$ be a partially ordered set. Let $f, g : X \to X$ be a pair of self-maps on $X$. $f$ is said to be $\preceq$-monotone if for any $x, y \in X$,
\[
(g(x), g(y)) \in E_\preceq \implies (f(x), f(y)) \in E_\preceq.
\]

**Theorem 3.2.3.** Let $(X, D, \preceq)$ be a complete partially ordered JS-metric space. Let $f, g : X \to X$ be a pair of self-maps on $X$ which satisfy the following conditions:

(i) $f(X) \subseteq g(X)$,

(ii) $f$ is $\preceq$-monotone,

(iii) there exists $x_0 \in X$ such that $(g(x_0), f(x_0)) \in E_\preceq$,

(iv) there exists $\beta \in F'$ such that if $(g(x), g(y)) \in E_\preceq$ then
\[
D(f(x), f(y)) \leq \beta(M(g(x), g(y)))M(g(x), g(y))
\]
where $M(g(x), g(y)) = \max\{D(g(x), g(y)), D(g(x), f(x)), D(g(y), f(y))\}$,

(v) $g$ is $D$-continuous,

(vi) $\sup\{D(g(x_0), f(y)) : y \in X\} < \infty$,

(vii) $f$ and $g$ are commuting.
(viii) $f$ is $D$-continuous.

Then there is $\omega \in X$ is a coincidence point of $f$ and $g$. Moreover, if $\omega' \in X$ such that $\omega' = f(\omega) = g(\omega)$ and $D(g(\omega), g(\omega')) < \infty$, then $\omega'$ is common fixed point of $f$ and $g$.

Proof. By assumption (iii) we have $x_0 \in X$ and by assumption (i), we can pick $x_1 \in X$ such that $g(x_1) = f(x_0)$. By assumption (i) again, we can pick $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continue the procedure to get a sequence $\{x_n\} \subset X$ such that

$$g(x_n) = f(x_{n-1})$$

for each $n \in \mathbb{N}$. (3.6)

If $g(x_{n_0}) = g(x_{n_0-1})$ for some $n_0 \in \mathbb{N}$, then $x_{n_0-1}$ is a coincidence point of $f$ and $g$. Therefore, in what follows, we will assume that for each $n \in \mathbb{N}$, $g(x_n) \neq g(x_{n-1})$ holds. By (iii), $(g(x_0), g(x_1)) \in E_\leq$, since $f$ is $\leq$-monotone, we have

$$(g(x_1), g(x_2)) \in E_\leq.$$ 

Continuing this method, we can show that for each $n \in \mathbb{N}$

$$(g(x_{n-1}), g(x_n)) \in E_\leq. \quad (3.7)$$

First, we will show that $\lim_{n \to \infty} D(g(x_n), g(x_{n+1})) = 0$. By contradiction, suppose

$$\lim_{n \to \infty} D(g(x_n), g(x_{n+1})) \neq 0,$$

that is, there exists $\epsilon > 0$ for which we can obtain subsequence $\{n_k\}$ such that $n_k \geq k$ and $\epsilon \leq D(g(x_{n_k}), g(x_{n_k+1}))$. Consider,

$$D(g(x_{n_k}), g(x_{n_k+1})) = D(f(x_{n_k-1}), f(x_{n_k})) \leq \beta(M(g(x_{n_k-1}), g(x_{n_k})))M(g(x_{n_k-1}), g(x_{n_k})), $$

where $M(g(x_{n_k-1}), g(x_{n_k})) = \max\{D(g(x_{n_k-1}), g(x_{n_k})), D(g(x_{n_k-1}), f(x_{n_k-1})), \}$

$$D(g(x_{n_k}), f(x_{n_k})).$$

Since $\beta \in [0, 1]$, $M(g(x_{n_k-1}), g(x_{n_k}) = D(g(x_{n_k-1}), g(x_{n_k})).$ Continuing this process, we get that

$$D(g(x_{n_k}), g(x_{n_k+1})) \leq \prod_{i=1}^{n_k} \beta(M(g(x_{n_k-i}), g(x_{n_k+1-i})))M(g(x_0), g(x_1)),$$

where $M(g(x_0), g(x_1)) = \max\{D(g(x_0), g(x_1)), D(g(x_0), f(x_0)), D(g(x_1), f(x_1))\}$. Since $\beta \in [0, 1]$, $M(g(x_0), g(x_1)) = D(g(x_0), f(x_0)) < \infty$. We choose $i_k$ such that

$$\beta(M(g(x_{n_k-i_k}), g(x_{n_k+1-i_k}))) := \max\{\beta(M(g(x_{n_k-i}), g(x_{n_k+1-i}))) : 1 \leq i \leq n_k\}.$$ 

Define $\eta := \limsup_{k \to \infty} \{\beta(M(g(x_{n_k-i_k}), g(x_{n_k+1-i_k})))\}$. 

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If \( \eta < 1 \), then \( \lim_{k \to \infty} \mathcal{D}(g(x_{n_k}), g(x_{n_k+1})) = 0 \), which contradicts to the assumption.

If \( \eta = 1 \), by passing through a subsequence, then we may assume that

\[
\lim_{k \to \infty} \beta(M(g(x_{n_k-i_k}), g(x_{n_k+1-i_k}))) = 1.
\]

Since \( \beta \in \mathcal{F} \), we have

\[
\lim_{k \to \infty} M(g(x_{n_k-i_k}), g(x_{n_k+1-i_k})) = 0,
\]

that is, there exists \( k_0 \in \mathbb{N} \) such that

\[
M(g(x_{n_{k_0-i_{k_0}}}), g(x_{n_{k_0+1-i_{k_0}}})) < \frac{\epsilon}{2}.
\]

So, we have

\[
\epsilon \leq \mathcal{D}(g(x_{n_{k_0}}), g(x_{n_{k_0}+1})) \\
\leq \prod_{j=1}^{i_{k_0}} \beta(M(g(x_{n_{k_0-j}}), g(x_{n_{k_0+1-j}})))M(g(x_{n_{k_0-i_{k_0}}}), g(x_{n_{k_0+1-i_{k_0}}})) \\
< \frac{\epsilon}{2},
\]

which is a contradiction. Therefore, we get that

\[
\lim_{n \to \infty} \mathcal{D}(g(x_n), g(x_{n+1})) = 0. \quad (3.8)
\]

Now, we show that \( \{g(x_n)\} \) is a \( \mathcal{D} \)-Cauchy sequence. By contradiction, suppose \( \{g(x_n)\} \) is not a \( \mathcal{D} \)-Cauchy sequence, that is, there exists \( \epsilon > 0 \) for which we can obtain subsequences \( \{n_k\}, \{m_k\} \) such that \( n_k, m_k \geq k \) and \( \epsilon \leq \mathcal{D}(g(x_{n_k}), g(x_{n_k+m_k})) \). Consider,

\[
\mathcal{D}(g(x_{n_k}), g(x_{n_k+m_k})) = \mathcal{D}(f(x_{n_k-1}), f(x_{n_k+m_k-1})) \\
\leq \beta(M(g(x_{n_k-1}), g(x_{n_k+m_k-1})))M(g(x_{n_k-1}), g(x_{n_k+m_k-1})),
\]

where \( M(g(x_{n_k-1}), g(x_{n_k+m_k-1})) = \max\{\mathcal{D}(g(x_{n_k-1}), g(x_{n_k+m_k-1})), \mathcal{D}(g(x_{n_k-1}), f(x_{n_k-1})), \mathcal{D}(g(x_{n_k+m_k-1}), f(x_{n_k+m_k-1}))\} \).

If \( M(g(x_{n_k-1}), g(x_{n_k+m_k-1})) = \mathcal{D}(g(x_{n_k-1}), f(x_{n_k-1})) \) or \( \mathcal{D}(g(x_{n_k+m_k-1}), f(x_{n_k+m_k-1})) \),

from equation (3.8), we get \( \lim_{k \to \infty} \mathcal{D}(g(x_{n_k}), g(x_{n_k+m_k})) = 0 \), which contradicts to the assumption. So \( M(g(x_{n_k-1}), g(x_{n_k+m_k-1})) = \mathcal{D}(g(x_{n_k-1}), g(x_{n_k+m_k-1})) \). Continuing this process, we get that

\[
\mathcal{D}(g(x_{n_k}), g(x_{n_k+m_k})) \leq \prod_{i=1}^{n_k} \beta(M(g(x_{n_k-i}), g(x_{n_k+m_k-i})))M(g(x_0), g(x_{m_k})).
\]
We choose $i_k$ such that
\[ \beta(M(g(x_{n_k-i_k}), g(x_{n_k+m_k-i_k}))) = \max \{ \beta(M(g(x_{n_k-i}), g(x_{n_k+m_k-i}))) : 1 \leq i \leq n_k \}. \]

Define $\eta := \limsup_{k \to \infty} \{ \beta(M(g(x_{n_k-i_k}), g(x_{n_k+m_k-i_k}))) \}.$

If $\eta < 1$, then $\lim_{k \to \infty} \mathcal{D}(g(x_{n_k}), g(x_{n_k+m_k})) = 0$, which contradicts the assumption.

If $\eta = 1$, by passing through a subsequence, then we may assume that
\[ \lim_{k \to \infty} \beta(M(g(x_{n_k-i_k}), g(x_{n_k+m_k-i_k}))) = 1. \]

Since $\beta \in \mathcal{F}$, we have
\[ \lim_{k \to \infty} M(g(x_{n_k-i_k}), g(x_{n_k+m_k-i_k})) = 0, \]
that is, there exists $k_0 \in \mathbb{N}$ such that
\[ M(g(x_{n_{k_0}-i_{k_0}}, g(x_{n_{k_0}+m_{k_0}-i_{k_0}})) < \frac{\epsilon}{2}. \]

So, we have
\[
\begin{align*}
\epsilon & \leq \mathcal{D}(g(x_{n_{k_0}}), g(x_{n_{k_0}+m_{k_0}})) \\
& \leq \prod_{j=1}^{i_{k_0}} \beta(M(g(x_{n_{k_0}-j}, g(x_{n_{k_0}+m_{k_0}-j}))) M(g(x_{n_{k_0}-i_{k_0}}, g(x_{n_{k_0}+m_{k_0}-i_{k_0}})) \\
& < \frac{\epsilon}{2},
\end{align*}
\]
which is a contradiction. Therefore $\{g(x_n)\}$ is a $\mathcal{D}$-Cauchy sequence. By $\mathcal{D}$-completeness of $(X, \mathcal{D})$, there exists some $\omega \in X$ such that
\[ \lim_{n \to \infty} \mathcal{D}(g(x_n), \omega) = 0. \quad (3.9) \]

By equation (3.6), then we get
\[ \lim_{n \to \infty} \mathcal{D}(f(x_{n-1}), \omega) = \lim_{n \to \infty} \mathcal{D}(f(x_n), \omega) = 0. \quad (3.10) \]

By the continuity of $g$, equations (3.9) and (3.10), we have
\[ \lim_{n \to \infty} \mathcal{D}(g(g(x_n)), g(\omega)) = 0, \quad (3.11) \]
and,
\[ \lim_{n \to \infty} \mathcal{D}(g(f(x_n)), g(\omega)) = 0. \]
Now, we show that $\omega$ is a coincidence point of $f$ and $g$. To accomplish this, we use assumption (viii). Suppose that $f$ is continuous. By (3.9), we obtain
\[
\lim_{n \to \infty} D(f(g(x_n)), f(\omega)) = 0.
\]
Since $f$ and $g$ are compatible, we have $f(\omega) = g(\omega)$. Therefore $\omega \in X$ is a coincidence point of $f$ and $g$.

Assume $\omega' \in X$ such that $\omega' = f(\omega) = g(\omega)$ and $D(g(\omega), g(\omega')) < \infty$. Since $f$ and $g$ are commuting, then $f(\omega') = f(g(\omega)) = g(f(\omega)) = g(\omega')$.

Consider $D(g(\omega), g(\omega')) \leq \beta(M(g(\omega), g(\omega')))M(g(\omega), g(\omega'))$,

where $M(g(\omega), g(\omega')) = \max\{D(g(\omega), g(\omega')), D(g(\omega), f(\omega)), D(g(\omega'), f(\omega'))\}$.

Since $D(g(\omega), g(\omega')) < \infty$, then $g(\omega) = g(\omega')$. Therefore $\omega' = f(\omega') = g(\omega')$.

Hence $\omega'$ is a common fixed point of $f$ and $g$.

**Example 3.2.4.** Let $X = [0, 1], D(x, y) = \max\{x, y\}$ for each $x, y \in X$. Then $(X, D, \leq)$ is a complete partially ordered JS-metric space. Let $f, g : X \to X$ be given by
\[
f(x) = \frac{x^4}{8} \quad \text{and} \quad g(x) = \frac{x^2}{2}
\]
for all $x \in X$.

Clearly, $f$ is $\leq$-monotone, $f$ and $g$ are $D$-continuous and $(0, 0) \in E_{\leq}$. Let $\beta : [0, +\infty] \to [0, 1)$ be a mapping defined by
\[
\beta(t) = \begin{cases} \frac{1}{1+t} & \text{if } t \in (0, \infty) \\ 0 & \text{if } t \in \{0, \infty\}. \end{cases}
\]

For $(g(x), g(y)) \in E_{\leq}$, we have $D(f(x), f(y)) \leq \beta(M(g(x), g(y)))M(g(x), g(y))$, where $M(g(x), g(y)) = \max\{D(g(x), g(y)), D(g(x), f(x)), D(g(y), f(y))\}$. Therefore, all the requirements of Theorem 3.2.3 are satisfied and $x = 0$ is a coincidence point of $f$ and $g$.

On setting $g$ is identity mapping, in Theorem 3.2.3, the following result appears:

**Corollary 3.2.5.** Let $(X, D, \leq)$ be a complete partially ordered JS-metric space. Let $f : X \to X$ be a self-map on $X$ which satisfies the following conditions:

(i) $f$ is $\leq$-monotone (i.e., for any $x, y \in X$ if $(x, y) \in E_{\leq}$ then $(f(x), f(y)) \in E_{\leq}$),

(ii) there exists $x_0 \in X$ such that $(x_0, f(x_0)) \in E_{\leq}$,

(iii) there exists $\beta \in F'$ such that if
\[
(x, y) \in E_{\leq} \quad \text{then} \quad D(f(x), f(y)) \leq \beta(M(x, y))M(x, y),
\]

where $M(x, y) = \max\{D(x, y), D(x, f(x)), D(y, f(y))\}$,
(iv) $\sup\{D(x_0, f(y)) : y \in X\} < \infty$,

(v) $f$ is $D$-continuous.

Then $f$ has a fixed point in $X$.

Here, we give an example to validate Corollary 3.2.5.

Example 3.2.6. Let $X = \{0, 1\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$, $D(x, y) = |x| + |y|$ for each $x, y \in X$. Then $(X, D, \leq)$ is a complete partially ordered JS-metric space. Let $f : X \to X$ be given by

$$f(x) = \begin{cases} 
  0 & \text{if } x = 0 \\
  \frac{1}{3n+1} & \text{if } x = \frac{1}{3} \text{ for all } n \in \mathbb{N} \\
  \frac{1}{3} & \text{if } x = 1.
\end{cases}$$

Clearly, $f$ is $\leq$-monotone, $D$-continuous and $(0, 0) \in E_\leq$. Define $\beta : [0, +\infty) \to [0, 1)$ by $\beta(t) = \frac{1}{t}$ for all $t \in [0, \infty]$. For $(x, y) \in E_\leq$, we have $D(f(x), f(y)) \leq \beta(M(x, y))M(x, y)$, where $M(x, y) = \max\{D(x, y), D(x, f(x)), D(y, f(y))\}$. Thus, all the requirements of Corollary 3.2.5 are satisfied and $x = 0$ is a fixed point in $X$.

Now, as an application of Theorem 3.2.3, using the techniques of Erhan et al. [12], we first introduce the following notation.

Definition 3.2.7. Let $(X, D)$ be a JS-metric space and define $\Delta_2 : X^2 \times X^2 \to [0, +\infty]$, for every $(x, y), (u, v) \in X^2$, by

$$\Delta_2((x, y), (u, v)) = \max\{D(x, u), D(y, v)\}.$$ 

And we define

$$((x, y), (u, v)) \in E_\leq \iff (x, u) \in E_\leq \text{ and } (y, v) \in E_\leq.$$ 

Given the partially ordered JS-metric space $(X, D, \leq)$, let us consider the partially ordered JS-metric space $(X^2, \Delta_2, \sqsubseteq)$, where $\Delta_2$ and $\sqsubseteq$ were defined in Definition 3.2.7. We define the mappings $T_F, T_g : X^2 \to X^2$, for every $(x, y) \in X^2$, by

$$T_F(x, y) = (F(x, y), F(y, x)) \quad \text{and} \quad T_g(x, y) = (g(x), g(y)).$$ 

Under these conditions, it is not difficult to obtain the following lemma:

Lemma 3.2.8. Let $(X, D, \leq)$ be a partially ordered JS-metric space, let $F : X \times X \to X$, $g : X \to X$ be a pair of mappings. Then the following statements hold:
(i) If \((X, D)\) is JS-metric space, then \((X^2, \Delta_2)\) is JS-metric space.

(ii) \((X, D)\) is \(D\)-complete if and only if \((X^2, \Delta_2)\) is \(\Delta_2\)-complete.

(iii) \(F(X^2) \subseteq g(X)\) is equivalent to \(T_F(X^2) \subseteq T_g(X^2)\).

(iv) If \(F\) is \(\preceq\)-\(g\)-monotone, then \(T_F\) is \(\sqsubseteq T_g\)-monotone.

(v) There exists \(x_0, y_0 \in X\) such that \((g(x_0), F(x_0, y_0)) \in E_{\preceq}\) and \((g(y_0), F(y_0, x_0)) \in E_{\preceq}\) is equivalent to the existence of a point \((x_0, y_0) \in X^2\) such that \((T_g(x_0, y_0), T_F(x_0, y_0)) \in E_{\preceq}\).

(vi) If there exists \(\beta \in F^\prime\) such that for every \(x, y, u, v \in X\) satisfying \((g(x), g(u)) \in E_{\preceq}\) and \((g(y), g(v)) \in E_{\preceq}\),

\[
\mathcal{D}(F(x, y), F(u, v)) \leq \beta(\max\{M(g(x), g(u)), M(g(y), g(v))\})
\]

\[
\max\{M(g(x), g(u)), M(g(y), g(v))\},
\]

where,

\[
\max\{M(g(x), g(u)), M(g(y), g(v))\} = \max\{\mathcal{D}(g(x), g(u)), \mathcal{D}(g(x), F(x, y)), \mathcal{D}(g(u), F(u, v)), \mathcal{D}(g(y), g(v)), \mathcal{D}(g(y), F(y, x)), \mathcal{D}(g(v), F(v, u))\}.
\]

Then

\[
\Delta_2(T_F(x, y), T_F(u, v)) \leq \beta(M(T_g(x, y), T_g(u, v)))M(T_g(x, y), T_g(u, v))
\]

where,

\[
M(T_g(x, y), T_g(u, v)) = \max\{\Delta_2(T_g(x, y), T_g(u, v)), \Delta_2(T_g(x, y), T_F(x, y)), \Delta_2(T_g(u, v), T_F(u, v))\}
\]

for every \((x, y), (u, v) \in X^2\) such that \((T_g(x, y), T_g(u, v)) \in E_{\preceq}\).

(vii) If \(g\) is \(D\)-continuous, then \(T_g\) is \(\Delta_2\)-continuous.

(viii) If \(\sup\{\mathcal{D}(g(x_0), F(x, y)), \mathcal{D}(g(y_0), F(y, x)) : x, y \in X\} < \infty\), then

\[
\sup\{\Delta_2(T_g(x_0, y_0), T_F(x, y)) : x, y \in X\} < \infty.
\]

(ix) If \(g\) and \(F\) are commuting, then \(T_g\) and \(T_F\) are commuting.

(x) If \(F\) is \(D\)-continuous, then \(T_F\) is \(\Delta_2\)-continuous.

(xii) There exist \((\omega, \omega') \in X^2\) is a coupled coincidence point of mappings \(g\) and \(F\) if and only if it is a coincidence point of \(T_g\) and \(T_F\).
Proof. Items (i), (ii), (iii), (v), (vii), (viii), (x) and (xii) are obvious.

(iiv) Assume that $F$ is $\preceq$-$g$-monotone, and let $(x, y)$ and $(u, v) \in X^2$ be such that $(T_g(x, y), T_g(u, v)) \in E_\succsim$. Then $(g(x), g(u)) \in E_{\preceq}$ and $(g(y), g(v)) \in E_{\preceq}$. Since $F$ is $\preceq$-$g$-monotone, we deduce that $(F(x, y), F(u, y)) \in E_{\preceq}$ and $(F(u, y), F(u, v)) \in E_{\preceq}$. Therefore

$$(F(x, y), F(u, v)) \in E_{\preceq}. \quad (3.12)$$

Similarly, we deduce that $(F(y, x), F(v, x)) \in E_{\preceq}$ and $(F(v, x), F(v, u)) \in E_{\preceq}$. Therefore

$$(F(y, x), F(v, u)) \in E_{\preceq}. \quad (3.13)$$

By (3.12) and (3.13), We have $(T_F(x, y), T_F(u, v)) \in E_\succsim$ and this means that $T_F$ is $\preceq$-$T_g$-monotone.

(vi) Assume that there exists $\beta \in F'$ such that for all $x, y, u, v \in X$ satisfying $(g(x), g(u)) \in E_{\preceq}$ and $(g(y), g(v)) \in E_{\preceq}$,

$$\mathcal{D}(F(x, y), F(u, v)) \leq \beta(\max\{M(g(x), g(u)), M(g(y), g(v))\})$$

where $\max\{M(g(x), g(u)), M(g(y), g(v))\} = \max\{\mathcal{D}(g(x), g(u)), \mathcal{D}(g(x), F(x, y)), \mathcal{D}(g(u), F(u, v)), \mathcal{D}(g(y), F(y, x)), \mathcal{D}(g(v), F(v, u))\}$. Let $(x, y)$ and $(u, v) \in X^2$ be such that $(T_g(x, y), T_g(u, v)) \in E_\succsim$. Therefore $(g(x), g(u)) \in E_{\preceq}$ and $(g(y), g(v)) \in E_{\preceq}$. Consider,

$$\Delta_2(T_F(x, y), T_F(u, v))$$

$$= \Delta_2((F(x, y), F(y, x)), (F(u, v), F(v, u)))$$

$$= \max\{\mathcal{D}(F(x, y), F(u, v)), \mathcal{D}(F(y, x), F(v, u))\}$$

$$\leq \max\{\beta(\max\{M(g(x), g(u)), M(g(y), g(v))\}) \max\{M(g(x), g(u)), M(g(y), g(v))\},$$

$$\beta(\max\{M(g(y), g(v)), M(g(x), g(u))\}) \max\{M(g(y), g(v)), M(g(x), g(u))\})\}$$

$$= \beta(\max\{M(g(x), g(u)), M(g(y), g(v))\}) \max\{M(g(x), g(u)), M(g(y), g(v))\})$$

$$= \beta(M(T_g(x, y), T_g(u, v))).$$

Therefore, $\Delta_2(T_F(x, y), T_F(u, v)) \leq \beta(M(T_g(x, y), T_g(u, v)))M(T_g(x, y), T_g(u, v))$,

where $M(T_g(x, y), T_g(u, v)) = \max\{\Delta_2(T_g(x, y), T_g(u, v)), \Delta_2(T_g(x, y), T_F(x, y)), \Delta_2(T_g(u, v), T_F(u, v))\}$ for all $(x, y), (u, v) \in X^2$ such that $(T_g(x, y), T_g(u, v)) \in E_\succsim$.

(ix) Assume that $g$ and $F$ are commuting and let $(x, y) \in X^2$. Then

$$T_g(T_F(x, y)) = T_g(F(x, y), F(y, x))$$
\[ g(F(x, y)), g(F(y, x)) \]
\[ = (F(g(x), g(y)), F(g(y), g(x))) \]
\[ = T_F(g(x), g(y)) \]
\[ = T_F(T_g(x, y)). \]

Therefore \( T_g \) and \( T_F \) are commuting.

Now, we show that the following result is a consequence of Lemma 3.2.8 and Theorem 3.2.3.

**Theorem 3.2.9.** Let \((X, D, \leq)\) be a complete partially ordered JS-metric space. Let \( F : X \times X \to X, g : X \to X \) be a pair of mappings which satisfy the following conditions:

(i) \( F(X^2) \subseteq g(X) \),

(ii) \( F \) is \( \leq \)-monotone,

(iii) there exist \( x_0, y_0 \in X \) such that \((g(x_0), F(x_0, y_0)) \in E_{\leq} \) and \((g(y_0), F(y_0, x_0)) \in E_{\leq} \),

(iv) there exists \( \beta \in \mathcal{F}' \) such that for all \( x, y, u, v \in X \) satisfying \((g(x), g(u)) \in E_{\leq} \) and \((g(y), g(v)) \in E_{\leq} \),

\[ D(F(x, y), F(u, v)) \leq \beta(\max\{M(g(x), g(u)), M(g(y), g(v))\}). \]

where, \( \max\{M(g(x), g(u)), M(g(y), g(v))\} = \max\{D(g(x), g(u)), D(g(x), F(x, y)), D(g(u), F(u, v)), D(g(y), g(v)), D(g(y), F(y, x)), D(g(v), F(v, u))\} \),

(v) \( g \) is \( D \)-continuous,

(vi) \( \sup\{D(g(x_0), F(x, y)), D(g(y_0), F(y, x)) : x, y \in X\} < \infty \),

(vii) \( g \) and \( F \) are commuting,

(viii) \( F \) is \( D \)-continuous.

Then \( g \) and \( F \) have a coupled coincidence point.

**Proof.** Apply Theorem 3.2.3 to the mapping \( f = T_F \) and \( g = T_g \) in the ordered metric space \((X^2, \Delta_2, \sqsubseteq)\), taking into account all items of Lemma 3.2.8. \( \square \)
3.3 Coupled Coincidence Point Results in Partially Ordered JS-Metric Spaces

Following Kadelburg et al. [19], let $\Theta'$ be the family of all functions $\theta : [0, +\infty] \times [0, +\infty] \to [0, 1)$ which satisfying the following conditions:

$(\theta'_1)$ $\theta(s, t) = \theta(t, s)$ for all $s, t \in [0, +\infty],$

$(\theta'_2)$ for any two sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative real numbers,

$$\lim_{n \to \infty} \theta(s_n, t_n) = 1 \text{ implies } \lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = 0.$$

**Example 3.3.1.** Let $\theta : [0, +\infty] \times [0, +\infty] \to [0, 1)$ be defined by

$$\theta(s, t) = \begin{cases} 
\ln(1 + \frac{s}{3} + \frac{t}{3}), & \text{if } s \in (0, \infty) \text{ or } t \in (0, \infty), \\
\frac{1}{2}, & \text{if } s = \infty \text{ or } t = \infty, \\
0, & \text{if } s = 0 \text{ and } t = 0.
\end{cases}$$

Then $\theta \in \Theta'.$

**Definition 3.3.2.** [19] Let $F : X \times X \to X, g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty].$ Then $F$ and $g$ are said to be $\alpha-$admissible if

$$\alpha((gx, gy), (gu, gv)) \geq 1 \text{ implies } \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$$

for all $x, y, u, v \in X.$

**Definition 3.3.3.** [19] Let $F : X \times X \to X, g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty].$ Then $F$ and $g$ are said to be triangular $\alpha-$admissible if $F$ and $g$ are $\alpha-$admissible and

$$\alpha((gx, gy), (gu, gv)) \geq 1 \text{ and } \alpha((gu, gv), (F(u, v), F(v, u))) \geq 1 \text{ imply }$$

$$\alpha((gx, gy), (F(u, v), F(v, u))) \geq 1$$

for all $x, y, u, v \in X.$

Now, we present our first main result as follows:

**Theorem 3.3.4.** Let $(X, D, \preceq)$ be a complete partially ordered JS-metric space, and let $F : X \times X \to X, g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty].$ Suppose that the following conditions hold:

(i) $F(X^2) \subseteq g(X),$
(ii) $F$ is $\preceq$-monotone and $D$-continuous,

(iii) $g$ is $D$-continuous, and commutes with $F$,

(iv) $g$ and $F$ are triangular $\alpha$-admissible,

(v) there exists $\theta \in \Theta'$ such that

\[
\alpha((gx, gy), (gu, gv))D(F(x, y), F(u, v)) \\
\leq \theta(D(gx, gu), D(gy, gv))M((gx, gu), (gy, gv)),
\]

where

\[
M((gx, gu), (gy, gv)) = \max\{D(gx, gu), D(gy, gv), D(gx, F(x, y)), \\
D(gy, F(y, x)), D(gu, F(u, v)), D(gv, F(v, u))\},
\]

for all $x, y, u, v \in X$ with $(gx, gu) \in E_\preceq$ and $(gy, gv) \in E_\preceq$,

(vi) there exist $x_0, y_0 \in X$ such that

\[
(gx_0, F(x_0, y_0)), (gy_0, F(y_0, x_0)) \in E_\preceq, \quad \text{and}
\]

\[
\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \quad \text{and}
\]

\[
\alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,
\]

(vii) if $\{x_n\}, \{y_n\}$ are sequences such that

\[
\lim_{n \to \infty} D(gx_n, gx_{n+1}) = 0 \quad \text{and} \quad \lim_{n \to \infty} D(gy_n, gy_{n+1}) = 0,
\]

then $\sup\{D(gx_0, gx_n), D(gy_0, gy_n) : n \in \mathbb{N}\} < \infty$.

Then $g$ and $F$ have a coupled coincidence point.

Proof. Let $x_0$ and $y_0$ be elements in $X$ satisfying assumption (vi). Since $F(X^2) \subseteq g(X)$, we can pick $x_1$ and $y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Since $F(X^2) \subseteq g(X)$ again, we can pick $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continue this procedure to obtain sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

\[
gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n)
\]

for each $n \in \mathbb{N}$.

If $gx_{n_0+1} = gx_{n_0}$ and $gy_{n_0+1} = gy_{n_0}$ for some $n_0 \in \mathbb{N}$, then $(x_{n_0}, y_{n_0})$ is a coupled coincidence point of $g$ and $F$. Therefore, in what follows, we will assume that for each $n \in \mathbb{N}$,

\[
gx_{n+1} \neq gx_n \quad \text{or} \quad gy_{n+1} \neq gy_n.
\]
By the condition \((vi)\),

\[(gx_0, gx_1) \in E_\preceq \text{ and } (gy_0, gy_1) \in E_\preceq.\]

Since \(F\) is \(\preceq\)-monotone,

\[(F(x_0, y_0), F(x_1, y_1)) \in E_\preceq \text{ and } (F(y_0, x_0), F(y_1, x_1)) \in E_\preceq.\]

That is,

\[(gx_1, gx_2) \in E_\preceq \text{ and } (gy_1, gy_2) \in E_\preceq.\]

Continuing this method, we get that

\[(gx_n, gx_{n+1}) \in E_\preceq \text{ and } (gy_n, gy_{n+1}) \in E_\preceq \text{ hold for all } n \in \mathbb{N}.\]

By transitivity of \(\preceq\),

\[(gx_n, gx_{n+m}) \in E_\preceq \text{ and } (gy_n, gy_{n+m}) \in E_\preceq \text{ for all } n, m \in \mathbb{N}.\]

By assumption \((vi)\),

\[\alpha((gx_0, gy_0), (gx_1, gy_1)) = \alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1.\]

Since \(F\) and \(g\) are \(\alpha\)-admissible, we obtain

\[\alpha((gx_1, gy_1), (gx_2, gy_2)) = \alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) \geq 1.\]

Thus, by mathematical induction, we have

\[\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1 \text{ for all } n \in \mathbb{N}.\]

Similarly,

\[\alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1 \text{ for all } n \in \mathbb{N}.\]

Since \(F\) and \(g\) are triangular \(\alpha\)-admissible,

\[\alpha((gx_n, gy_n), (gx_{n+m}, gy_{n+m})) \geq 1 \text{ and }\]
\[\alpha((gy_n, gx_n), (gy_{n+m}, gx_{n+m})) \geq 1 \text{ for all } n \in \mathbb{N}.\]

Now, we will show that

\[\lim_{n \to \infty} D(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n \to \infty} D(gy_n, gy_{n+1}) = 0.\]
By way of contradiction, suppose at least one of \( \lim_{n \to \infty} \mathcal{D}(gx_n, gx_{n+1}) \neq 0 \) or \( \lim_{n \to \infty} \mathcal{D}(gy_n, gy_{n+1}) \neq 0 \) holds. Then there exists \( \varepsilon > 0 \) for which we can obtain subsequence \( \{n_k\} \) such that \( n_k \geq k \) and
\[
\varepsilon \leq \max\{\mathcal{D}(gx_{n_k}, gx_{n_k+1}), \mathcal{D}(gy_{n_k}, gy_{n_k+1})\}.
\]
Consider
\[
\begin{align*}
\mathcal{D}(gx_{n_k}, gx_{n_k+1}) &= \mathcal{D}(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k}, y_{n_k})) \\
&\leq \alpha((gx_{n_k-1}, y_{n_k-1}),(gx_{n_k}, y_{n_k})) \mathcal{D}(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k}, y_{n_k})) \\
&\leq \theta(\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k}))M((gx_{n_k-1}, gx_{n_k}),(gy_{n_k-1}, gy_{n_k}))
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{D}(gy_{n_k}, gy_{n_k+1}) &= \mathcal{D}(F(y_{n_k-1}, x_{n_k-1}), F(y_{n_k}, x_{n_k})) \\
&\leq \alpha((gy_{n_k-1}, x_{n_k-1}),(gy_{n_k}, x_{n_k})) \mathcal{D}(F(y_{n_k-1}, x_{n_k-1}), F(y_{n_k}, x_{n_k})) \\
&\leq \theta(\mathcal{D}(gy_{n_k-1}, gy_{n_k}), \mathcal{D}(gx_{n_k-1}, gx_{n_k}))M((gy_{n_k-1}, gy_{n_k}),(gx_{n_k-1}, gx_{n_k})).
\end{align*}
\]
Since \( \theta(s, t) \in [0, 1) \) for all \( s, t \in [0, +\infty] \),
\[
M((gx_{n_k-1}, gx_{n_k}),(gy_{n_k-1}, gy_{n_k})) = M((gy_{n_k-1}, gy_{n_k}), \mathcal{D}(gx_{n_k-1}, gx_{n_k}))
\]
\[
= \max\{\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k})\}.
\]
From (3.14), (3.15) and (3.16),
\[
\max\{\mathcal{D}(gx_{n_k}, gx_{n_k+1}), \mathcal{D}(gy_{n_k}, gy_{n_k+1})\} \leq \theta(\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k})) \max\{\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k})\}.
\]
Continuing this process, we get that
\[
\max\{\mathcal{D}(gx_{n_k}, gx_{n_k+1}), \mathcal{D}(gy_{n_k}, gy_{n_k+1})\} \leq \prod_{i=1}^{n_k} \theta(\mathcal{D}(gx_{n_k-i}, gx_{n_k+1-i}), \mathcal{D}(gy_{n_k-i}, gy_{n_k+1-i})) \max\{\mathcal{D}(gx_0, gx_1), \mathcal{D}(gy_0, gy_1)\}.
\]
We choose \( i_k \) such that
\[
\theta(\mathcal{D}(gx_{n_k-i_k}, gx_{n_k+1-i_k}), \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+1-i_k})) = \max_{1 \leq i \leq n_k} \{\theta(\mathcal{D}(gx_{n_k-i}, gx_{n_k+1-i}), \mathcal{D}(gy_{n_k-i}, gy_{n_k+1-i}))\}.
\]
\[35\]
Define \( \eta := \limsup_{k \to \infty} \{ \theta(\mathcal{D}(g_{x_{n_k}}-i_k, g_{x_{n_k}+1-i_k}), \mathcal{D}(g_{y_{n_k}}-i_k, g_{y_{n_k}+1-i_k})) \} \).

If \( \eta < 1 \), then
\[
\lim_{k \to \infty} \max_{i, j} \{ \mathcal{D}(g_{x_{n_k}}, g_{x_{n_k}+1}), \mathcal{D}(g_{y_{n_k}}, g_{y_{n_k}+1}) \} = 0,
\]
which contradicts the assumption.

If \( \eta = 1 \), by passing through a subsequence, then we may assume that
\[
\lim_{k \to \infty} \theta(\mathcal{D}(g_{x_{n_k}}-i_k, g_{x_{n_k}+1-i_k}), \mathcal{D}(g_{y_{n_k}}-i_k, g_{y_{n_k}+1-i_k})) = 1.
\]
Since \( \theta \in \Theta' \), we have
\[
\lim_{k \to \infty} \mathcal{D}(g_{x_{n_k}}-i_k, g_{x_{n_k}+1-i_k}) = 0 \quad \text{and} \quad \lim_{k \to \infty} \mathcal{D}(g_{y_{n_k}}-i_k, g_{y_{n_k}+1-i_k}) = 0.
\]
That is, there exists \( k_0 \in \mathbb{N} \) such that
\[
\mathcal{D}(g_{x_{nk_0}-i_k}, g_{x_{nk_0}+1-i_k}) < \frac{\epsilon}{2} \quad \text{and} \quad \mathcal{D}(g_{yn_k_0-i_k}, g_{yn_k_0+1-i_k}) < \frac{\epsilon}{2}.
\]
Thus, we have
\[
\epsilon \leq \max_{i, j} \left\{ \mathcal{D}(g_{x_{nk_0}}, g_{x_{nk_0}+1}), \mathcal{D}(g_{yn_k_0}, g_{yn_k_0+1}) \right\} \\
\leq \prod_{j=1}^{k_0} \theta(\mathcal{D}(g_{x_{nk_0}-j}, g_{x_{nk_0}+1-j}), \mathcal{D}(g_{yn_k_0}-j, g_{yn_k_0+1-j})) \\
\max \left\{ \mathcal{D}(g_{x_{nk_0}-i_k}, g_{x_{nk_0}+1-i_k}), \mathcal{D}(g_{yn_k_0-i_k}, g_{yn_k_0+1-i_k}) \right\} \\
< \frac{\epsilon}{2},
\]
which is a contradiction. Therefore,
\[
\lim_{n \to \infty} \mathcal{D}(g_n, g_{n+1}) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathcal{D}(y_n, y_{n+1}) = 0. \tag{3.17}
\]

Next, we will show that \( \{g_n\} \) and \( \{y_n\} \) are \( \mathcal{D} \)-Cauchy sequences. By contradiction, suppose at least one of \( \{g_n\} \) or \( \{y_n\} \) is not a \( \mathcal{D} \)-Cauchy sequence. Then there exists \( \epsilon' > 0 \) for which we can obtain subsequences \( \{n_k\} \), \( \{m_k\} \) such that \( n_k, m_k \geq k \) and
\[
\epsilon' \leq \max \{ \mathcal{D}(g_{x_{nk}}, g_{x_{nk}+m_k}), \mathcal{D}(g_{yn_k}, g_{yn_k+m_k}) \}.
\]
Consider
\[
\mathcal{D}(g_{x_{nk}}, g_{x_{nk}+m_k}) \\
= \mathcal{D}(F(x_{nk-1}, y_{nk-1}), F(x_{nk+m_k-1}, y_{nk+m_k-1})) \\
\leq \alpha\left( (g_{x_{nk-1}}, g_{yn_k}), (g_{x_{nk}+m_k-1}, g_{yn_k+m_k-1}) \right) \\
\mathcal{D}(F(x_{nk-1}, y_{nk-1}), F(x_{nk+m_k-1}, y_{nk+m_k-1})) \\
\leq \theta(\mathcal{D}(g_{x_{nk-1}}, g_{x_{nk+m_k-1}}), \mathcal{D}(g_{yn_k}, g_{yn_k+m_k-1}))
\]
\[ M((g x_{nk-1}, g x_{nk+m_k-1}), (g y_{nk-1}, g y_{nk+m_k-1})) \]  

(3.18)

and

\[
\mathcal{D}(g y_{nk}, g y_{nk+m_k}) = \mathcal{D}(F(y_{nk}-1, x_{nk}-1), F(y_{nk+m_k-1}, x_{nk+m_k}-1))
\]

\[
\leq \alpha((g y_{nk-1}, g x_{nk-1}), (g y_{nk+m_k-1}, g x_{nk+m_k-1}))
\]

\[
\mathcal{D}(F(y_{nk}-1, x_{nk}-1), F(y_{nk+m_k-1}, x_{nk+m_k}-1))
\]  

\[
\leq \theta(\mathcal{D}(g y_{nk-1}, g y_{nk+m_k-1}), \mathcal{D}(g x_{nk-1}, g x_{nk+m_k-1}))
\]

\[
M((g y_{nk-1}, g y_{nk+m_k-1}), (g x_{nk-1}, g x_{nk+m_k-1}))
\]  

(3.19)

From (3.17),

\[
M((g x_{nk-1}, g x_{nk+m_k-1}), (g y_{nk-1}, g y_{nk+m_k-1}))
\]

\[
= M((g y_{nk-1}, g y_{nk+m_k-1}), (g x_{nk-1}, g x_{nk+m_k-1}))
\]

\[
= \max\{\mathcal{D}(g x_{nk-1}, g x_{nk+m_k-1}), \mathcal{D}(g y_{nk-1}, g y_{nk+m_k-1})\}
\]  

(3.20)

From (3.18), (3.19) and (3.20),

\[
\max\{\mathcal{D}(g x_{nk}, g x_{nk+m_k}), \mathcal{D}(g y_{nk}, g y_{nk+m_k})\}
\]

\[
\leq \theta(\mathcal{D}(g x_{nk-1}, g x_{nk+m_k-1}), \mathcal{D}(g y_{nk-1}, g y_{nk+m_k-1}))
\]

\[
\max\{\mathcal{D}(g x_{nk-1}, g x_{nk+m_k-1}), \mathcal{D}(g y_{nk-1}, g y_{nk+m_k-1})\}.
\]

Continuing this process, we get that

\[
\max\{\mathcal{D}(g x_{nk}, g x_{nk+m_k}), \mathcal{D}(g y_{nk}, g y_{nk+m_k})\}
\]

\[
\leq \prod_{i=1}^{n_k} \theta(\mathcal{D}(g x_{nk-i}, g x_{nk+m_k-i}), \mathcal{D}(g y_{nk-i}, g y_{nk+m_k-i}))
\]

\[
\max\{\mathcal{D}(g x_0, g x_m), \mathcal{D}(g y_0, g y_m)\}.
\]

We choose \( i_k \) such that

\[
\theta(\mathcal{D}(g x_{nk-i_k}, g x_{nk+m_k-i_k}), \mathcal{D}(g y_{nk-i_k}, g y_{nk+m_k-i_k}))
\]

\[
= \max_{1 \leq i \leq n_k} \{\theta(\mathcal{D}(g x_{nk-i}, g x_{nk+m_k-i}), \mathcal{D}(g y_{nk-i}, g y_{nk+m_k-i}))\}.
\]

Define \( \eta := \limsup_{k \to \infty} \theta(\mathcal{D}(g x_{nk-i_k}, g x_{nk+m_k-i_k}), \mathcal{D}(g y_{nk-i_k}, g y_{nk+m_k-i_k})) \).

If \( \eta < 1 \), then \( \lim_{k \to \infty} \max\{\mathcal{D}(g x_{nk}, g x_{nk+m_k}), \mathcal{D}(g y_{nk}, g y_{nk+m_k})\} = 0 \), which contradicts to
the assumption.

If \( \eta = 1 \), by passing through a subsequence, then we may assume that

\[
\lim_{k \to \infty} \theta(D(gx_{n_k} - i_k, gx_{n_k + m_k - i_k}), D(gy_{n_k} - i_k, gy_{n_k + m_k - i_k})) = 1.
\]

Since \( \theta \in \Theta' \), we have

\[
\lim_{k \to \infty} D(gx_{n_k} - i_k, gx_{n_k + m_k - i_k}) = 0 \quad \text{and} \quad \lim_{k \to \infty} D(gy_{n_k} - i_k, gy_{n_k + m_k - i_k}) = 0.
\]

Then there exists \( k_0 \in \mathbb{N} \) such that

\[
D(gx_{n_{k_0} - i_{k_0}}, gx_{n_{k_0} + m_{k_0} - i_{k_0}}) < \frac{\epsilon'}{2} \quad \text{and} \quad D(gy_{n_{k_0} - i_{k_0}}, gy_{n_{k_0} + m_{k_0} - i_{k_0}}) < \frac{\epsilon'}{2}.
\]

Thus, we have

\[
\epsilon' \leq \max\{D(gx_{n_{k_0}}, gx_{n_{k_0} + m_{k_0}}), D(gy_{n_{k_0}}, gy_{n_{k_0} + m_{k_0}})\}
\leq \prod_{j=1}^{i_{k_0}} \theta(D(gx_{n_{k_0} - j}, gx_{n_{k_0} + m_{k_0} - j}), D(gy_{n_{k_0} - j}, gy_{n_{k_0} + m_{k_0} - j}))
\max\{D(gx_{n_{k_0} - i_{k_0}}, gx_{n_{k_0} + m_{k_0} - i_{k_0}}), D(gy_{n_{k_0} - i_{k_0}}, gy_{n_{k_0} + m_{k_0} - i_{k_0}})\}
< \frac{\epsilon'}{2},
\]

which is a contradiction. Therefore, \( \{gx_n\} \) and \( \{gy_n\} \) are \( D \)-Cauchy sequences. By the completeness of \( (X, D) \), there exist some \( \omega, \omega' \in X \) such that

\[
\lim_{n \to \infty} D(F(x_n, y_n), \omega) = \lim_{n \to \infty} D(gx_n, \omega) = 0 \quad \text{and} \quad \lim_{n \to \infty} D(F(y_n, x_n), \omega') = \lim_{n \to \infty} D(gy_n, \omega') = 0.
\]

By the continuity of \( g \),

\[
\lim_{n \to \infty} D(g(F(x_n, y_n)), g\omega) = 0 \quad \text{and} \quad \lim_{n \to \infty} D(g(F(y_n, x_n)), g\omega') = 0.
\]

By the continuity of \( F \),

\[
\lim_{n \to \infty} D(F(gx_n, gy_n), F(\omega, \omega')) = 0 \quad \text{and} \quad \lim_{n \to \infty} D(F(gy_n, gx_n), F(\omega', \omega)) = 0.
\]

By the commuting of \( g \) and \( F \) and the uniqueness of the limit, \( g\omega = F(\omega, \omega') \) and \( g(\omega') = F(\omega', \omega) \). Therefore, \( (\omega, \omega') \in X \times X \) is a coupled coincidence point of \( g \) and \( F \).

In our second main result, we obtain a coupled coincidence result for \( \alpha \)-Geraghty contraction type in JS-metric spaces.
Theorem 3.3.5. Let \((X, \mathcal{D}, \preceq)\) be a complete partially ordered JS-metric space, and let \(F : X \times X \to X\), \(g : X \to X\) and \(\alpha : X^2 \times X^2 \to [0, +\infty]\). Suppose that the following conditions hold:

(i) \(F(X^2) \subseteq g(X)\),

(ii) \(F\) is \(\preceq\)-monotone,

(iii) \(g\) is \(\mathcal{D}\)-continuous, and commutes with \(F\),

(iv) \(g\) and \(F\) are triangular \(\alpha\)-admissible,

(v) there exists \(\beta \in F\) such that for all \(x, y, u, v \in X\) satisfying \((gx, gu) \in E_\preceq\) and \((gy, gv) \in E_\preceq\),

\[
\alpha((gx, gy), (gu, gv))D(F(x, y), F(u, v)) \\
\leq \beta(M((gx, gu), (gy, gv)))M((gx, gu), (gy, gv)),
\]

where

\[
M((gx, gu), (gy, gv)) = \max\{D(gx, gu), D(gy, gv), D(gx, F(x, y)), D(gy, F(y, x))\},
\]

(vi) there exist \(x_0, y_0 \in X\) such that

\[
(gx_0, F(x_0, y_0)), (gy_0, F(y_0, x_0)) \in E_\preceq,
\]

\[
\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \quad \text{and} \quad \alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,
\]

(vii) if \(\{x_n\}, \{y_n\}\) are sequences such that

\[
\lim_{n \to \infty} D(gx_n, gx_{n+1}) = 0 \quad \text{and} \quad \lim_{n \to \infty} D(gy_n, gy_{n+1}) = 0,
\]

then \(\sup\{D(gx_0, gx_n), D(gy_0, gy_n) : n \in \mathbb{N}\} < \infty\),

(viii) (a) \(F\) is \(\mathcal{D}\)-continuous or (b) for \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(X\) such that

\[
(gx_n, gx_{n+1}), (gy_n, gy_{n+1}) \in E_\preceq,
\]

\[
\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1,
\]

\[
\alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1 \quad \text{for all } n \in \mathbb{N}
\]
and
\[ \lim_{n \to \infty} \mathcal{D}(gx_n, \omega) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathcal{D}(gy_n, \omega') = 0, \]
we have
\[ (ggx_n, g\omega), (ggy_n, g\omega') \in E_{\leq}, \]
\[ \alpha((ggx_n, ggy_n), (g\omega, g\omega')) \geq 1, \]
\[ \alpha((ggy_n, ggx_n), (g\omega', g\omega)) \geq 1 \quad \text{for all } n \in \mathbb{N}. \]

Then \( g \) and \( F \) have a coupled coincidence point.

**Proof.** Using similar idea as the proof of Theorem 3.3.4, we can construct Cauchy sequences \( \{gx_n\} \) and \( \{gy_n\} \) in a complete JS-metric space \((X, \mathcal{D})\). Then there exist \( \omega, \omega' \in X \) such that

\[
\begin{align*}
\lim_{n \to \infty} \mathcal{D}(F(x_n, y_n), \omega) &= \lim_{n \to \infty} \mathcal{D}(gx_n, \omega) = 0 \quad \text{and} \\
\lim_{n \to \infty} \mathcal{D}(F(y_n, x_n), \omega') &= \lim_{n \to \infty} \mathcal{D}(gy_n, \omega') = 0.
\end{align*}
\]

By the continuity of \( g \), we obtain
\[
\begin{align*}
\lim_{n \to \infty} \mathcal{D}(gF(x_n, y_n), g\omega) &= \lim_{n \to \infty} \mathcal{D}(ggx_n, g\omega) = 0 \quad \text{and} \\
\lim_{n \to \infty} \mathcal{D}(gF(y_n, x_n), g\omega') &= \lim_{n \to \infty} \mathcal{D}(ggy_n, g\omega') = 0.
\end{align*}
\]

If \( F \) is \( \mathcal{D} \)-continuous, it is easy to show that \( g \) and \( F \) have a coupled coincidence point. Otherwise, By assumptions (v) and (vii), we have

\[ \mathcal{D}(F(gx_n, gy_n), F(\omega, \omega')) \leq \alpha((ggx_n, ggy_n), (g\omega, g\omega')) \mathcal{D}(F(gx_n, gy_n), F(\omega, \omega')) \]
\[ \leq \beta(M((ggx_n, g\omega), (ggy_n, g\omega'))M((ggx_n, g\omega), (ggy_n, g\omega'))) \]
\[ \leq \beta(M((ggx_n, g\omega), (ggy_n, g\omega'))M((ggx_n, g\omega), (ggy_n, g\omega'))) \]
\[ \quad (3.21) \]

and

\[ \mathcal{D}(F(gy_n, gx_n), F(\omega', \omega)) \leq \alpha((ggy_n, ggx_n), (g\omega', g\omega)) \mathcal{D}(F(gy_n, gx_n), F(\omega', \omega)) \]
\[ \leq \beta(M((ggy_n, g\omega'), (ggx_n, g\omega))M((ggy_n, g\omega'), (ggx_n, g\omega))) \]
\[ \quad (3.22) \]

where

\[ M((ggx_n, g\omega), (ggy_n, g\omega')) \]
Suppose \( g\omega \neq F(\omega, \omega') \) or \( g\omega' \neq F(\omega', \omega) \), that is,
\[
D := \max \{ D(g\omega, F(\omega, \omega')), D(g\omega', F(\omega', \omega)) \} > 0.
\]
Letting \( n \to \infty \) in (3.23), we have
\[
\lim_{n \to \infty} M((g\omega_n, g\omega'), (gy_n, g\omega')) = D.
\] (3.24)

From (3.21) and (3.22), we have
\[
\max \{ M((g\omega_n, g\omega'), (gy_n, g\omega')) \}
\]
Taking limit on both sides of above inequalities, we have
\[
\lim_{n \to \infty} \beta(M((g\omega_n, g\omega), (gy_n, g\omega'))) = 1.
\]
This implies \( \lim_{n \to \infty} M((g\omega_n, g\omega), (gy_n, g\omega')) = 0 \) which contradicts to equation (3.24).

Therefore, \( g\omega = F(\omega, \omega') \) and \( g\omega' = F(\omega', \omega) \), that is, \( (\omega, \omega') \in X \times X \) is a coupled coincidence point of \( g \) and \( F \).

**Example 3.3.6.** Let \( X = [0, +\infty] \), \( D(x, y) = \max\{x, y\} \) for all \( x, y \in X \). Then \( (X, D, \leq) \) is a complete partially ordered JS-metric space. Define mappings \( F : X \times X \to X \) and \( g : X \to X \) by
\[
F(x, y) = \begin{cases} 
\frac{x+y}{6}, & \text{if } x, y \in [0, +\infty), \\
+\infty, & \text{otherwise},
\end{cases} \quad g(x) = \begin{cases} 
2x, & \text{if } x \in [0, +\infty), \\
+\infty, & \text{otherwise}.
\end{cases}
\]
A mapping \( \alpha : X^2 \times X^2 \to [0, +\infty] \) is given by
\[
\alpha((x, y), (u, v)) = \begin{cases} 
1, & \text{if } x \leq y \text{ and } u \leq v, \\
0, & \text{otherwise}.
\end{cases}
\]
Let \( x \leq u \) and \( y \leq v \). If \( x > y \) or \( u > v \), then it is obvious that the assumption \((v)\) of Theorem 3.3.4 holds. Otherwise,
\[
\alpha((gx, gy), (gu, gv))D(F(x, y), F(u, v)) = \max\{\frac{x+y}{6}, \frac{u+v}{6}\} = \frac{u+v}{6} \leq \frac{2v}{6}
\]
\[
\leq \frac{4v}{6} = \frac{2v}{3}.
\]
Thus, the assumption (v) of Theorem 3.3.4 holds for $\theta(s, t) = \frac{1}{6}$ for all $s, t \in [0, +\infty]$. We can easily check that all conditions of Theorem 3.3.4 hold. Therefore, $g$ and $F$ have coupled coincidence point which is $(0, 0)$. However, we cannot apply Theorem 2.5.9 to show the existence of a coupled coincidence point for $g$ and $F$. 

$$= \frac{1}{6}M((gx, gu), (gy, gv)).$$
CHAPTER 4

Conclusion

In this chapter, we conclude all main results of the thesis. It is organized by dividing into three sections:

4.1 Fixed Point Theorems for Generalized R-Contraction in b-Metric Spaces

In this thesis, we have introduced generalized R-contraction conditions on a complete b-Metric spaces as:

Let \((X; d)\) be a complete b-metric space with coefficient \(K \geq 1\). Let \(T : X \to X\) be \(R'\)-contraction with respect to \(r \in R^+\). If \(r(Kt, s) \leq s - Kt\) for all \(s, t \in (0, \infty)\) then \(T\) has a unique fixed point.

4.2 Coincidence Point Theorems for Geraghty-type Contraction Mappings in JS-Metric Spaces

Next, we have our main result which successively guarantee a coincidence point as follows:

(1) Let \((X, D, \preceq)\) be a complete partially ordered JS-metric space. Let \(f, g : X \to X\) be a pair of self-maps on \(X\) which satisfy the following conditions:

(i) \(f(X) \subseteq g(X)\),

(ii) \(f\) is \(\preceq\)-\(g\)-monotone,

(iii) there exists \(x_0 \in X\) such that \((g(x_0), f(x_0)) \in E_\preceq\),

(iv) there exists \(\beta \in \mathcal{F}'\) such that if \((g(x), g(y)) \in E_\preceq\) then

\[
D(f(x), f(y)) \leq \beta(M(g(x), g(y))) M(g(x), g(y))
\]

where \(M(g(x), g(y)) = \max\{D(g(x), g(y)), D(g(x), f(x)), D(g(y), f(y))\}\),

(v) \(g\) is \(D\)-continuous,
\( (vi) \sup \{D(g(x_0), f(y)) : y \in X \} < \infty, \)

\( (vii) \) \( f \) and \( g \) are commuting,

\( (viii) \) \( f \) is \( D \)-continuous.

Then \( \omega \in X \) is a coincidence point of \( f \) and \( g \). Moreover, if \( \omega' \in X \) such that \( \omega' = f(\omega) = g(\omega) \) and \( D(g(\omega), g(\omega')) < \infty \), then \( \omega' \) is a common fixed point of \( f \) and \( g \).

The following result is a consequence of coincidence point theorem:

\( (2) \) Let \( (X, D, \leq) \) be a complete partially ordered JS-metric space. Let \( F : X \times X \to X \), \( g : X \to X \) be a pair of mappings which satisfy the following conditions:

\( (i) \) \( F(X^2) \subseteq g(X) \),

\( (ii) \) \( F \) is \( \leq \)-monotone,

\( (iii) \) there exist \( x_0, y_0 \in X \) such that \( (g(x_0), F(x_0, y_0)) \in E_\leq \) and \( (g(y_0), F(y_0, x_0)) \in E_\leq \),

\( (iv) \) there exists \( \beta \in F' \) such that for all \( x, y, u, v \in X \) satisfying \( (g(x), g(u)) \in E_\leq \) and \( (g(y), g(v)) \in E_\leq \),

\[ D(F(x, y), F(u, v)) \leq \beta(\max\{M(g(x), g(u)), M(g(y), g(v))\}) \]

\[ \max\{M(g(x), g(u)), M(g(y), g(v))\}, \]

where,

\[ \max\{M(g(x), g(u)), M(g(y), g(v))\} \]

\[ \max\{D(g(x), g(u)), D(g(x), F(x, y)), D(g(u), F(u, v)), D(g(y), F(y, x)), D(g(y), F(v, u))\}, \]

\( (v) \) \( g \) is \( D \)-continuous,

\( (vi) \) \( \sup \{D(g(x_0), F(x, y)), D(g(y_0), F(y, x)) : x, y \in X \} < \infty \),

\( (vii) \) \( g \) and \( F \) are commuting,

\( (viii) \) \( F \) is \( D \)-continuous.

Then \( g \) and \( F \) have a coupled coincidence point.
4.3 Coupled Coincidence Point Results in Partially Ordered JS-Metric Spaces

Finally, we have results of coupled coincidence point under $\alpha$-Geraghty contraction type in a complete partially ordered JS-metric spaces as follows:

(1) Let $(X, \mathcal{D}, \preceq)$ be a complete partially ordered JS-metric space, and let $F : X \times X \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty]$. Suppose that the following conditions hold:

(i) $F(X^2) \subseteq g(X)$,

(ii) $F$ is $\preceq$-$g$-monotone and $\mathcal{D}$-continuous,

(iii) $g$ is $\mathcal{D}$-continuous, and commutes with $F$,

(iv) $g$ and $F$ are triangular $\alpha$-admissible,

(v) there exists $\theta \in \Theta'$ such that

\[
\alpha((gx, gy), (gu, gv))\mathcal{D}(F(x, y), F(u, v)) \\
\leq \theta(\mathcal{D}(gx, gu), \mathcal{D}(gy, gv))M((gx, gu), (gy, gv)),
\]

where

\[
M((gx, gu), (gy, gv)) = \max\{\mathcal{D}(gx, gu), \mathcal{D}(gy, gv), \mathcal{D}(gx, F(x, y)), \mathcal{D}(gy, F(y, x)), \mathcal{D}(gu, F(u, v)), \mathcal{D}(gv, F(v, u))\},
\]

for all $x, y, u, v \in X$ with $(gx, gu) \in E_{\preceq}$ and $(gy, gv) \in E_{\preceq}$,

(vi) there exist $x_0, y_0 \in X$ such that

\[
(x_0, F(x_0, y_0), (y_0, F(y_0, x_0)) \in E_{\preceq}, \text{ and }
\]

\[
\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \text{ and }
\]

\[
\alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,
\]

(vii) if $\{x_n\}$ and $\{y_n\}$ are sequences such that

\[
\lim_{n \to \infty} \mathcal{D}(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n \to \infty} \mathcal{D}(gy_n, gy_{n+1}) = 0,
\]

then $\sup\{\mathcal{D}(gx_0, gx_n), \mathcal{D}(gy_0, gy_n) : n \in \mathbb{N}\} < \infty$.

Then $g$ and $F$ have a coupled coincidence point.
(2) Let \((X, \mathcal{D}, \succeq)\) be a complete partially ordered JS-metric space, and let \(F : X \times X \to X\), \(g : X \to X\) and \(\alpha : X^2 \times X^2 \to [0, +\infty]\). Suppose that the following conditions hold:

(i) \(F(X^2) \subseteq g(X)\),

(ii) \(F\) is \(\succeq\)-monotone,

(iii) \(g\) is \(\mathcal{D}\)-continuous, and commutes with \(F\),

(iv) \(g\) and \(F\) are triangular \(\alpha\)-admissible,

(v) there exists \(\beta \in \mathcal{F}\) such that for all \(x, y, u, v \in X\) satisfying \((gx, gu) \in E\succeq\) and \((gy, gv) \in E\succeq\),

\[
\alpha((gx, gy),(gu, gv)) \mathcal{D}(F(x, y), F(u, v)) \\
\leq \beta(M((gx, gu), (gy, gv))) M((gx, gu), (gy, gv)),
\]

where

\[
M((gx, gu), (gy, gv)) = \max\{\mathcal{D}(gx, gu), \mathcal{D}(gy, gv), \mathcal{D}(gx, F(x, y)), \mathcal{D}(gy, F(y, x)), \mathcal{D}(gu, F(u, v)), \mathcal{D}(gv, F(v, u))\},
\]

(vi) there exist \(x_0, y_0 \in X\) such that

\[
(gx_0, F(x_0, y_0)), (gy_0, F(y_0, x_0)) \in E\succeq,
\]

\[
\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \quad \text{and} \quad \alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,
\]

(vii) if \(\{x_n\}\) and \(\{y_n\}\) are sequences such that

\[
\lim_{n \to \infty} \mathcal{D}(gx_n, gx_{n+1}) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathcal{D}(gy_n, gy_{n+1}) = 0,
\]

then \(\sup\{\mathcal{D}(gx_0, gx_n), \mathcal{D}(gy_0, gy_n) : n \in \mathbb{N}\} < \infty\),

(viii) (a) \(F\) is \(\mathcal{D}\)-continuous or (b) for \(\{x_n\}\) and \(\{y_n\}\) are sequences in \(X\) such that

\[
(gx_n, gx_{n+1}), (gy_n, gy_{n+1}) \in E\succeq,
\]

\[
\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1,
\]

\[
\alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1 \quad \text{for all} \quad N
\]

and

\[
\lim_{n \to \infty} \mathcal{D}(gx_n, \omega) = 0 \quad \text{and} \quad \lim_{n \to \infty} \mathcal{D}(gy_n, \omega') = 0,
\]
we have

\[(ggx_n, g\omega), (ggy_n, g\omega') \in E \leq,\]

\[\alpha((ggx_n, ggy_n), (g\omega, g\omega')) \geq 1,\]

\[\alpha((ggy_n, ggx_n), (g\omega', g\omega)) \geq 1 \text{ for all } N.\]

Then \(g\) and \(F\) have a coupled coincidence point.
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